Gravitational lensing and superluminal effects by Kerr and Kerr-de Sitter black holes.

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Abstract

1 Introduction .

The gravitational bending of light (and the associate phenomenon of gravitational lensing) has been instrumental in unravelling the nature of the gravitational field and its cosmological implications.

- – Despite the importance of the gravitational bending of light not many exact analytic results for the $deflection\ angle$ from the gravitational field of important astrophysical objects are known in the literature.
  – One of the most important physical objects for Astrophysics and Gravitational Astronomy is the Kerr black hole.
  – The full analytic treatment of the Kerr and Kerr-de Sitter black holes as *gravitational lenses* was imperative since the Kerr black hole acts as a very strong gravitational lens and we may probe General Relativity, through the phenomenon of the bending of light induced by the spacetime curvature of a spinning black hole, at the $strong\ gravitational\ field\ regime$.
  – The cosmological constant Λ is the prime culprit responcible for the force that dominates the current observed expansion of the Universe and triggered the onset of the accelerated expansion shortly before the formation of the Solar system. It had been debated as to whether or not the cosmological constant bends light. Indeed, as we shall see today Λ does contribute to the gravitational bending of light.
  – In this talk I am happy to report my recent results in obtaining the full analytic solution for the deflection angle of generic light orbits in Kerr-de Sitter black hole spacetime, as well as the analytic treatment of the more involved issue of treating the Kerr and Kerr-de Sitter black holes as gravitational lenses G. V. Kraniotis, Clas.Quantum Grav. 28(2011) 085021.

2 Null geodesics in Kerr and Kerr-de Sitter spacetime.

Taking into account the contribution from the cosmological constant, Λ, the generalization of the Kerr solution is described by the Kerr-de Sitter metric element which in Boyer-Lindquist (BL) coordinates is given by [Stuchlik Calvani,Gen.R.Grav23 (1991),Demianski(1973)ActaAstron.23, Carter,Com.M.P.(1968)]:

$$
\begin{aligned}
\mathrm{d}s^2 = {} & \frac{\Delta_r}{\Xi^2\rho^2}(c\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2 - \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 \\
& -\frac{\Delta_\theta\sin^2\theta}{\Xi^2\rho^2}(ac\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2
\end{aligned} \tag{1}
$$

$$
\Delta_\theta := 1 + \frac{a^2\Lambda}{3}\cos^2\theta,\ \ \Xi := 1 + \frac{a^2\Lambda}{3},\ \ \rho^2 = r^2 + a^2\cos^2\theta, \tag{2}
$$

$$
\Delta_r := \left(1 - \frac{\Lambda}{3}r^2\right)(r^2+a^2) - 2\frac{GM}{c^2}r \tag{3}
$$

(rdel)

We denote by a the rotation (Kerr) parameter and M denotes the mass of the spinning black hole.

The relevant null geodesic differential equations for the calculation of the gravitational lensing effects (lens-equation) and for the calculation of the deflection angle are:

Kraniotis,CQG22(2005)4391–4424

$$
\int^r \frac{\mathrm{d}r}{\sqrt{R}} = \pm\int^\theta \frac{\mathrm{d}\theta}{\sqrt{\Theta}} \tag{4}
$$

and Tr2

$$
\Delta\phi = \int \mathrm{d}\phi = \int^\theta -\frac{\Xi^2}{\Delta_\theta\sin^2\theta}\frac{(a\sin^2\theta - \Phi)\mathrm{d}\theta}{\sqrt{\Theta}} + \int^r \frac{a\Xi^2}{\Delta_r}[(r^2+a^2) - a\Phi]\frac{\mathrm{d}r}{\sqrt{R}} \tag{5}
$$

(lens),(ray)

where

$$
R := \left\{\Xi^2\left[(r^2+a^2) - a\Phi\right]^2 - \Delta_r\left[\Xi^2(\Phi - a)^2 + \mathcal{Q}\right]\right\} \tag{6}
$$

and

$$
\Theta := \left\{[\mathcal{Q} + (\Phi - a)^2\Xi^2]\Delta_\theta - \frac{\Xi^2(a\sin^2\theta - \Phi)^2}{\sin^2\theta}\right\} \tag{7}
$$

We also derive the equation related to time-delay:

$$
ct = \int^r \frac{\Xi^2(r^2+a^2)\left[(r^2+a^2) - \Phi a\right]}{\pm\Delta_r\sqrt{R}}\mathrm{d}r - \int^\theta \frac{a\Xi^2(a\sin^2\theta - \Phi)}{\pm\Delta_\theta\sqrt{\Theta}}\mathrm{d}\theta. \tag{8}
$$

The parameters Φ, $\mathcal{Q}$ are associated to the first integrals of motion. The former is the impact parameter and the latter is related to the hidden first integral (due to the separation of variables in the corresponding Hamilton-Jacobi PDE).

We now turn our attention to the issue of treating the Kerr and Kerr-de Sitter black holes as gravitational lenses, construct the resulting geometry and lens equations and solve the latter in closed analytic form. In addition, we shall derive for the first time the solutions in closed analytic form for the magnification factors.

Previous efforts on the issue of gravitational lensing from a Kerr black hole were concentrated on various approximations and numerical techniques Bray I., *Phys.Rev.*D. **34** (1986) 367;S.E. Vazquez and E.P. Esteban, Nuov.Cim. 119B(2004) 489, M. Sereno and F. De Luca, *Phys.Rev.*D. **74** (2006) 123009, arXiv:astro-ph/0609435v2; V. Bozza, F. De Luca and G. Scarpetta, *Phys.Rev.*D. **74** (2006) 063001; V. Bozza, *Phys.Rev.*D. **78** (2008) 063014.

3 The Kerr black hole as a gravitational lens

Assume without loss of generality that the observer's position is at $(r_O, \theta_O, 0)$. Likewise, for the source we have (r_S, θ_S, ϕ_S) . In the observer's reference frame, an incoming light ray is described by a parametric curve $x(r), y(r), z(r)$, where $r^2 = x^2 + y^2 + z^2$. For large r this the usual radial BL coordinate.

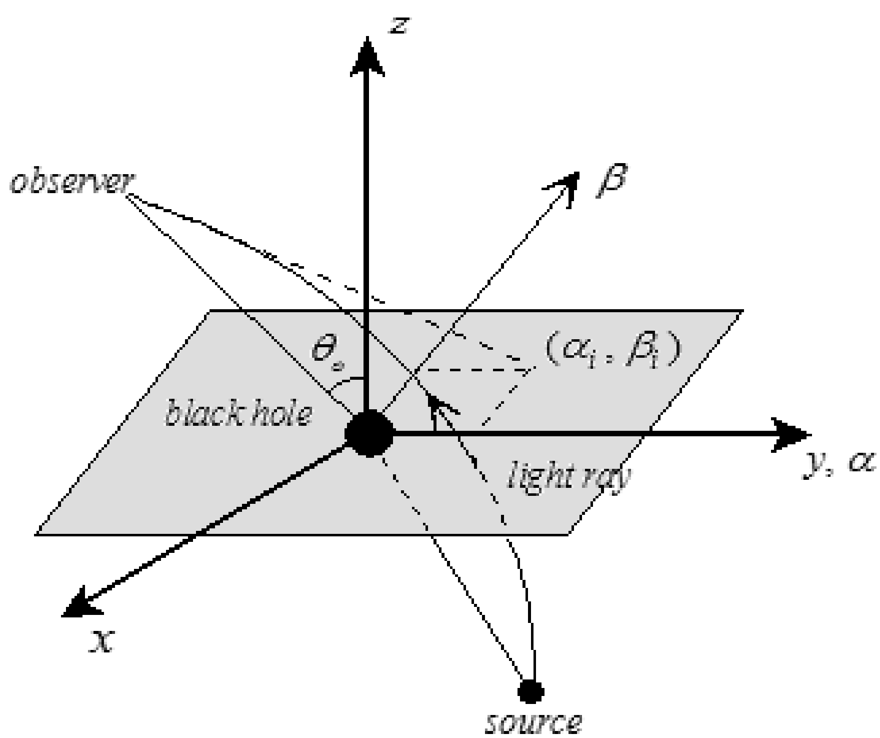
At the location of the observer, the tangent vector to the parametric curve is given by: $(\mathrm{d}x/\mathrm{d}r)|_{r_O}\widehat{\mathbf{x}} + (\mathrm{d}y/\mathrm{d}r)|_{r_O}\widehat{\mathbf{y}} + (\mathrm{d}z/\mathrm{d}r)|_{r_O}\widehat{\mathbf{z}}$. This vector describes a straight line which intersects the (α, β) plane or *observer's image plane* as it is usually called at (α_i, β_i). The point (α_i, β_i) is the point $(-\beta_i \cos\theta_O, \alpha_i, \beta_i \cos\theta_O)$ in the (x, y, z) system.

Our purpose now is to relate the α_i, β_i variables to the first integrals of motion $\Phi, \mathcal{Q}$. For this we need to use the equation of straight line in space. A straight line can be defined from a point $P_1(x_1, y_1, z_1)$ on it and a vector $\bar{\epsilon}(\epsilon_1, \epsilon_2, \epsilon_3)$ parallel to it. The analytic equations of straight line are then:

$$\frac{x - x_1}{\epsilon_1} = \frac{y - y_1}{\epsilon_2} = \frac{z - z_1}{\epsilon_3} \tag{9}$$

Applying (9) we derive the equations:

$$\frac{-\beta_i \cos\theta_O - r_O \sin\theta_O}{r_O \cos\theta_O \frac{d\theta}{dr}|_{r=r_0} + \sin\theta_O} = \frac{\alpha_i}{r_O \sin\theta_O \frac{d\phi}{dr}|_{r=r_O}} = \frac{\beta_i \cos\theta_O - r_O \cos\theta_O}{\cos\theta_O - r_O \sin\theta_O \frac{d\theta}{dr}|_{r=r_O}} \tag{10}$$


z
observer
β
θ$_o$
(α_i, β_i)
black hole
y, α
light ray
x
source

Solving for α_i, β_i we obtain the equations:

$$\alpha_i = -r_O^2 \sin\theta_O \frac{d\phi}{dr}|_{r=r_O} \tag{11}$$

$$\beta_i = r_O^2 \frac{d\theta}{dr}|_{r=r_O} \tag{12}$$

Now we have from the null geodesics that:

$$\frac{d\theta}{dr}|_{r=r_O} = \frac{\Theta(\theta_O)^{1/2}}{R(r_O)^{1/2}} \tag{13}$$

and

$$\frac{d\phi}{dr}|_{r=r_O} = \frac{\Phi}{\sqrt[2]{R(r_O)}} \frac{1}{\sin^2(\theta_O)} + \frac{2aGM\frac{r_O}{c^2} - a^2\Phi}{r_O^2\left[1+\frac{a^2}{r_O^2} - \frac{2GM}{r_O c^2}\right]} \frac{1}{\sqrt[2]{R(r_O)}} \tag{14}$$

Using eqns(13),(14) and assuming large observer's distance r_O(i.e. $r_O \longrightarrow \infty$) we derive simplified expressions relating the coordinates (α_i, β_i) on the observer's image plane to the integrals of motion:

$$\boxed{\Phi \simeq -\alpha_i \sin\theta_O} \tag{15}$$

$$\boxed{\mathcal{Q} \simeq \beta_i^2 + (\alpha_i^2 - a^2)\cos^2(\theta_O)} \tag{16}$$

(IP)

We can also express the position of the source on the observer's sky in terms of its coordinates (r_S, θ_S, ϕ_S) and the observer coordinates. Indeed, the equation for a straight line can be determined by two points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \tag{17}$$

Thus applying the above formula for the straight line connecting the observer and the source yields the equations:

$$\boxed{\alpha_S = \frac{r_O r_S \sin\theta_S \sin\phi_S}{r_O - r_S(\cos\theta_S\cos\theta_O + \sin\theta_O\sin\theta_S\cos\phi_S)}} \tag{18}$$

$$\boxed{\beta_S = \frac{-r_O r_S(\sin\theta_O\cos\theta_S - \sin\theta_S\cos\phi_S\cos\theta_O)}{r_O - r_S(\cos\theta_S\cos\theta_O + \sin\theta_O\sin\theta_S\cos\phi_S)}} \tag{19}$$

4 Magnification factors and positions of images.

The flux of an image of an infinitesimal source is the product of its surface brightness and the solid angle $\Delta\omega$ it subtends on the sky. Since the former quantity is unchanged during light deflection, the ratio of the flux of a sufficiently small image to that of its corresponding source in the absence of the lens, is given by

$$\mu = \frac{\Delta\omega}{(\Delta\omega)_0} = \frac{1}{|J|} \tag{20}$$

where 0-subscripts denote undeflected quantities and J is the Jacobian of the transformation $(x_{S,}\ y_S) \rightarrow (x_i, y_i)$ [1]. Writting $x_S = x_S(x_i, y_i), y_S = y_S(x_i, y_i)$ we can find expressions for the partial derivatives appearing in the Jacobian by differentiating equations (4) and (5). Indeed, the Jacobian is given by the expression:

$$J = xw - zy \tag{21}$$

where we defined: $x := \frac{\partial x_S}{\partial x_i}, y := \frac{\partial x_S}{\partial y_i}, z := \frac{\partial y_S}{\partial x_i}, w := \frac{\partial y_S}{\partial y_i}$. Writting equations (4) and (5) as follows(pvsol2) :

$$\begin{aligned} R_1(x_i, y_i) - A_1(x_i, y_i, x_S, y_S, m) &= 0 \\ \Delta\phi(x_S, y_S, n) - R_2(x_i, y_i) - A_2(x_i, y_i, x_S, y_S, m) &= 0 \end{aligned} \tag{22}$$

(wpf) (Aij)we set up the following system of equations:

$$\beta_1 = -\alpha_1 x - \alpha_2 z \tag{23}$$

$$\beta_2 = -\alpha_1 y - \alpha_2 w \tag{24}$$

$$-\beta_3 = \alpha_3 x + \alpha_4 z \tag{25}$$

$$-\beta_4 = \alpha_3 y + \alpha_4 w \tag{26}$$

where $\alpha_1 = \frac{\partial A_1}{\partial x_S},\ \alpha_2 = \frac{\partial A_1}{\partial y_S},\ \alpha_3 = -\frac{\partial \phi_s}{\partial x_S} - \frac{\partial A_2}{\partial x_S},\ \alpha_4 = -\frac{\partial \phi_s}{\partial y_S} - \frac{\partial A_2}{\partial y_S}$,

$\beta_1 = \frac{\partial R_1}{\partial x_i} - \frac{\partial A_1}{\partial x_i},\ \beta_2 = \frac{\partial R_1}{\partial y_i} - \frac{\partial A_1}{\partial y_i},\ \beta_3 = \frac{\partial R_2}{\partial x_i} + \frac{\partial A_2}{\partial x_i},\ \beta_4 = \frac{\partial R_2}{\partial y_i} + \frac{\partial A_2}{\partial y_i}$.

Solving for x, y, z, w we obtain:

(Magnification)

$$\mu = \frac{1}{|J|} = \left|\frac{\alpha_1\alpha_4 - \alpha_2\alpha_3}{\beta_1\beta_4 - \beta_2\beta_3}\right| \tag{27}$$

[1] Recall in the small angles approxiamation: $\alpha_i \approx r_O\ x_i,\ \beta_i \approx r_O\ y_i$. Also we define: $x_S := \frac{\alpha_S}{r_O}, y_S := \frac{\beta_S}{r_O}$.

5 The boundary of the shadow of the Kerr black hole.

The condition for a photon to escape to infinity , which is also the condition for the spherical photon orbits in Kerr spacetime, is given by the vanishing of the quartic polynomial $R(r)$ and its first derivative (also in this case $\frac{d^2R}{dr^2}|_{r=r_f} > 0$). Implementing these two conditions, expressions for the parameter Φ and Carter's constant $\mathcal{Q}$ are obtained:

$$\boxed{\Phi = \frac{a^2\frac{GM}{c^2}+a^2 r-3\frac{GM}{c^2}r^2+r^3}{a\left(\frac{GM}{c^2}-r\right)}, \qquad \mathcal{Q}=-\frac{r^3\left(-4a^2\frac{GM}{c^2}+r\left(\frac{-3GM}{c^2}+r\right)^2\right)}{a^2\left(\frac{GM}{c^2}-r\right)^2},}$$

The perturbed, from the radius $r = r_{\rm inst}$ of unstable spherical null orbits in Kerr spacetime, and thus escaped photon, will be detected on the observer's image plane, at the coordinates:

$$x_i = \frac{a^2(r+\frac{GM}{c^2})+r^2(r-\frac{3GM}{c^2})}{r_O\sin\theta_O a\left(r-\frac{GM}{c^2}\right)},$$
$$y_i = \frac{\pm\sqrt{-r^3[r(r-\frac{3GM}{c^2})^2-4a^2\frac{GM}{c^2}]-2a^2r(2a^2\frac{GM}{c^2}+r^3-3r\frac{G^2M^2}{c^4})z_O-a^4(r-\frac{GM}{c^2})^2z_O^2}}{r_O\sin\theta_O a\left(r-\frac{GM}{c^2}\right)} \tag{28}$$

A photon will be detected when the argument of the square root in 28 is positive. Also, $z_O := \cos^2\theta_O$. In Kraniotis, CQG 28 (2011) 085021 various constraints for the motion of light from the allowed polar region: $\theta_{\rm min} \leq \theta_S, \theta_O \leq \theta_{\rm max}$ were derived ($z_m \geq z_O$ etc).

6 Closed form solution for the angular integrals.

In this case we have to take into account the turning points in the polar coordinate. A generic angular polar integral can be written:

$$\pm\int_{\theta_1}^{\theta_2} = \int_{\min(z_1,z_2)}^{\max(z_1,z_2)} + [1-{\rm sign}(\theta_1\circ\theta_2)]\int_0^{\min(z_1,z_2)} \tag{29}$$

where:

$$\theta_1\circ\theta_2 := \cos\theta_1\cos\theta_2 \tag{30}$$

Indeed, using the variable $z := \cos^2\theta$ we derive:

$$-\frac{1}{2}\frac{\mathrm{d}z}{\sqrt{z}}\frac{1}{\sqrt{1-z}} = \mathrm{sign}(\frac{\pi}{2}-\theta)\mathrm{d}\theta \tag{31}$$

This is the result of the fact that in the interval $0 \leq \theta \leq \frac{\pi}{2}$, $\cos\theta \geq 0$ and $\sin\theta \geq 0$, while in the interval $\frac{\pi}{2} \leq \theta \leq \pi$, $\sin\theta \geq 0$, $\cos\theta \leq 0$.

Now, for a light trajectory that encounters m turning points ($m \geq 1$) we have:

$$\int^{\theta} = \pm\int_{\theta_S}^{\theta_{\mathrm{min/max}}} \underbrace{\pm\int_{\theta_{\mathrm{min/max}}}^{\theta_{\mathrm{max/min}}} \pm \int_{\theta_{\mathrm{max/min}}}^{\theta_{\mathrm{min/max}}}\cdots}_{m-1\text{ times}} \pm\int_{\theta_{\mathrm{max/min}}}^{\theta_O} = \tag{32}$$

$$\begin{aligned} &= \int_{z_S}^{z_m} + [1-\mathrm{sign}(\theta_S \circ \theta_{mS})]\int_0^{z_S} \\ &\quad + \int_{z_O}^{z_m} + [1-\mathrm{sign}(\theta_O \circ \theta_{mO})]\int_0^{z_O} \\ &\quad + 2(m-1)\int_0^{z_m} \end{aligned} \tag{33}$$

The roots z_m, z_3 (of $\Theta(\theta) = 0$) are expressed in terms of the integrals of motion and the cosmological constant by the expressions:

$$z_{m,3} = \frac{\mathcal{Q}+\Phi^2\Xi^2-H^2 \pm \sqrt{(\mathcal{Q}+\Phi^2\Xi^2-H^2)^2+4H^2\mathcal{Q}}}{-2H^2} \tag{34}$$

and(Lk)

$$H^2 := \frac{a^2\Lambda}{3}[\mathcal{Q}+(\Phi-a)^2\Xi^2] + a^2\Xi^2 \tag{35}$$

For $\Lambda = 0$, the turning points take the form:

$$z_m = \frac{a^2-\mathcal{Q}-\Phi^2+\sqrt{4a^2\mathcal{Q}+(-a^2+\mathcal{Q}+\Phi^2)^2}}{2a^2}, \tag{36}$$

where the subscript "m" stands for "min/max". The corresponding angles are:

$$\theta_{\mathrm{min/max}} = \mathrm{Arccos}(\pm\sqrt{z_m}) \tag{37}$$

.

Also:

$$\theta_{mO} := \mathrm{Arccos}(\mathrm{sign}(y_i)\sqrt{z_m}), \tag{38}$$

and

$$\theta_{mS} := \begin{cases} \theta_{mO}, & m\,\text{odd} \\ \pi - \theta_{mO}, & m\,\text{even} \end{cases}. \tag{39}$$

(mt)
Now for θ_j and $\theta_{\min\,/\,\max}$ in the same hemisphere:

$$\int_{\theta_j}^{\theta_{\min\,/\,\max}} \frac{\mathrm{d}\theta}{\pm\sqrt[2]{\Theta(\theta)}} = \frac{1}{2|a|}\int_{z_j}^{z_m} \frac{\mathrm{d}z}{\sqrt[2]{z(z_m-z)(z-z_3)}} \equiv I_3 \tag{40}$$

Let us now calculate the elliptic integral in eqn.(40) in closed analytic form. Applying the transformation:

$$z = z_m + \xi^2(z_j - z_m) \tag{41}$$

our integral is calculated in closed form in terms of
Appell's generalized hypergeometric function F_1 of two variables:

$$I_3 = \frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_j)}}{\sqrt[2]{z_m(z_m-z_3)}} F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_j}{z_m},\frac{z_m-z_j}{z_m-z_3}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \tag{42}$$

The function $F_1(\alpha,\beta,\beta',\gamma,x,y)$ is the first of the four Appell's hypergeometric functions of two variables x,y (Appell,1882, J.Math.Pures Appl.Liouville 8,173-216),

$$F_1(\alpha,\beta,\beta',\gamma,x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)}x^m y^n \tag{43}$$

which admits the following integral representation:

$$\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}du = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_1(\alpha,\beta,\beta',\gamma,x,y) \tag{44}$$

The double series converges when:

$$|x|<1, \quad |y|<1 \tag{45}$$

The above *Euler* integral representation is valid for: $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\gamma-\alpha)>0$. *Also* $\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx$ denotes the Gamma function. The Pochhammer symbol $(\alpha)_m=(\alpha,m)$ is defined as:

$$(\alpha)_m=\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}=\begin{cases}1, & \text{if} \quad m=0\\ \alpha(\alpha+1)\cdots(\alpha+m-1) & \text{if } m=1,2,3\end{cases} \tag{46}$$

On the other hand using the transformation:

$$\boxed{z=\frac{uz_jz_m-z_jz_m}{uz_j-z_m}} \tag{47}$$

we calculate in closed form:

$$\begin{aligned}&\frac{1}{2\,|a|}\int_0^{z_j}\frac{\mathrm{d}z}{\sqrt[2]{z(z_m-z)(z-z_3)}}\\ &=\frac{1}{|a|}\frac{\sqrt[2]{z_j}}{z_m}\sqrt[2]{\frac{z_j-z_m}{z_3-z_j}}F_1\left(1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j}{z_m},\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)}\right)\\ &=\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)}}}{\sqrt[2]{z_m-z_3}}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)},\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)}\right)\end{aligned} \tag{48}$$

In going from the second line to the third of (48) we made use of the following identity of Appell's first generalised hypergeometric function of two variables:

$$F_1(\alpha,\beta,\beta',\gamma,x,y)=(1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}F_1(\gamma-\alpha,\beta,\gamma-\beta-\beta',\gamma,\frac{x-y}{x-1},y) \tag{49}$$

Likewise we derive the closed form solution for the following integral:

$$\begin{aligned}&\frac{1}{2|a|}\int_0^{z_j}\frac{\mathrm{d}z}{(1-z)\sqrt[2]{z(z_m-z)(z-z_3)}}\\ &=\frac{z_j}{z_m}\frac{1}{|a|}\frac{z_j-z_m}{1-z_j}\frac{1}{\sqrt[2]{z_j(z_j-z_m)(z_3-z_j)}}\times\\ &F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j(1-z_m)}{z_m(1-z_j)},\frac{z_j}{z_m},\frac{z_j(z_m-z_3)}{z_m(z_j-z_3)}\right)\\ &=\frac{1}{|a|}\frac{z_j}{z_m}\sqrt[2]{\frac{z_m}{-z_3z_j}}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},z_j,\frac{z_j}{z_m},\frac{z_j}{z_3}\right)\end{aligned} \tag{50}$$

Producing the last line of equation (50) we used the following formula for the Lauricella function F_D (FD):

Proposition 1

$$F_D(\alpha,\beta,\beta',\beta'',\gamma,x,y,z) = (1-y)^{\gamma-\alpha-\beta'}(1-x)^{-\beta}(1-z)^{-\beta''} \times F_D\left(\gamma-\alpha,\beta,\gamma-\beta-\beta'-\beta'',\beta'',\gamma,\frac{x-y}{x-1},y,\frac{z-y}{z-1}\right)$$

Proof. Applying the transformation:

$$u = \frac{1-\nu}{1-\nu y} \tag{51}$$

onto the integral:

$$IR_{F_D} = \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-u\ y)^{-\beta'}(1-uz)^{-\beta''}\mathrm{d}u \tag{52}$$

we derive:

$$(1-u)^{\gamma-\alpha-1} = \left(\frac{\nu(1-y)}{1-\nu y}\right)^{\gamma-\alpha-1}, \quad (1-ux)^{-\beta} = \left(\frac{(1-x)[1-\frac{\nu(x-y)}{(x-1)}]}{1-\nu y}\right)^{-\beta}$$

$$(1-u\ y)^{-\beta'} = \frac{(1-y)^{-\beta'}}{(1-\nu y)^{-\beta'}}, \quad (1-uz)^{-\beta''} = \left(\frac{(1-z)[1-\frac{\nu(z-y)}{z-1}]}{1-\nu y}\right)^{-\beta''} \tag{53}$$

and thus we obtain the result:

$$IR_{F_D} = (1-y)^{\gamma-\alpha}(1-x)^{-\beta}(1-y)^{-\beta'}(1-z)^{-\beta''} \times \int_0^1 \mathrm{d}\nu\ \nu^{\gamma-\alpha-1}(1-\nu)^{\alpha-1}(1-\nu\ y)^{-(\gamma-\beta-\beta'-\beta'')}(1-\nu\frac{x-y}{x-1})^{-\beta}(1-\nu\frac{z-y}{z-1})^{-\beta''} \tag{54}$$

or

$$F_D(\alpha,\beta,\beta',\beta'',\gamma,x,y,z) = (1-y)^{\gamma-\alpha-\beta'}(1-x)^{-\beta}(1-z)^{-\beta''} \times F_D\left(\gamma-\alpha,\beta,\gamma-\beta-\beta'-\beta'',\beta'',\gamma,\frac{x-y}{x-1},y,\frac{z-y}{z-1}\right)$$

■

Likewise:

$$I_4 := \frac{-\Phi}{2|a|}\int_{z_j}^{z_m} \frac{\mathrm{d}z}{(1-z)\sqrt[2]{z(z_m-z)(z-z_3)}} = \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_j)}{z_m}}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_j-z_m}{1-z_m},\frac{z_m-z_j}{z_m},\frac{z_m-z_j}{z_m-z_3}\right) \tag{55}$$

Let us see for instance the term in (39):

$$\pm\int_{\theta_{\min\,/\,\max}}^{\theta_{\max\,/\,\min}} = 2\int_0^{z_m} \tag{56}$$

since $\cos^2\theta_{\min\,/\,\max} = z_m$ and $\theta_{\min}\circ\theta_{\max} = -z_m$.(angul.)

Equation (55) for $z_j = 0$, becomes (tpL):

$$\begin{aligned}
&-\frac{\Phi}{2|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{-z_m}{1-z_m},1,\frac{z_m}{z_m-z_3}\right)\\
&=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{1}{(1-z_m)}\frac{\pi}{2}F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{-z_m}{1-z_m},\frac{z_m}{z_m-z_3}\right)\\
&=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{\pi}{2}F_1\left(\frac{1}{2},1,-\frac{1}{2},1,\frac{z_m(1-z_3)}{z_m-z_3},\frac{z_m}{z_m-z_3}\right)\\
&=-\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{\pi}{2}\frac{1}{1-z_3}\left(F(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3})-z_3F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{z_m(1-z_3)}{z_m-z_3},\frac{z_m}{z_m-z_3}\right)\right)
\end{aligned} \tag{57}$$

On the other hand the angular integrals of the form $\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}}$ in equation (5) are solved in closed analytic form as follows:

$$\begin{aligned}
\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}} &= \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\\
&+[1-\mathrm{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\frac{z_S}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt[2]{z_S(z_S-z_m)(z_3-z_S)}}\times\\
&F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)
\end{aligned} \tag{58}$$

(isim)An equivalent expression for the above integral is(F):

$$\begin{aligned}
\pm\int_{\theta_S}^{\theta_{\min\,/\,\max}} &= \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_S-z_m}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\\
&+[1-\mathrm{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\sqrt[2]{\frac{z_S}{z_m}}\sqrt[2]{\frac{z_m-z_3}{z_S-z_3}}\frac{1}{\sqrt[2]{z_m-z_3}}\times\\
&F_D\left(\frac{1}{2},1,\frac{1}{2},-\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_3)}{z_S-z_3},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)\\
&= \frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_S-z_m}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\\
&+[1-\mathrm{sign}(\theta_S\circ\theta_{ms})](-)\frac{\Phi}{|a|}\sqrt[2]{\frac{z_S}{z_m}}\sqrt[2]{\frac{z_m-z_3}{z_S-z_3}}\frac{1}{\sqrt[2]{z_m-z_3}}\times\\
&[\frac{-z_3}{1-z_3}F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_3)}{z_S-z_3},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)+\\
&\frac{1}{1-z_3}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right)]
\end{aligned}\tag{59}$$

Thus we have that (2len):

$$A_2(x_i, y_i, x_{S,}y_S, m) = 2(m-1) \times \left[-\frac{\Phi}{|a|} \frac{1}{\sqrt[2]{(z_m - z_3)}} \frac{1}{(1-z_m)} \frac{\pi}{2} F_1\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{-z_m}{1-z_m}, \frac{z_m}{z_m - z_3}\right)\right]$$
$$+\frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m - z_S)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1-z_m)} \times$$
$$F_D\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S - z_m}{1-z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3}\right)$$
$$+[1 - \mathrm{sign}(\theta_S \circ \theta_{ms})](-)\frac{\Phi}{|a|} \frac{z_S}{z_m} \frac{z_S - z_m}{1 - z_S} \frac{1}{\sqrt[2]{z_S(z_S - z_m)(z_3 - z_S)}} \times$$
$$F_D\left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S(1-z_m)}{z_m(1-z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_{3)}}\right) +$$
$$+\frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m - z_O)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1-z_m)} \times$$
$$F_D\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_O - z_m}{1-z_m}, \frac{z_m - z_O}{z_m}, \frac{z_m - z_O}{z_m - z_3}\right)$$
$$[1 - \mathrm{sign}(\theta_O \circ \theta_{mO})](-)\frac{\Phi}{|a|} \frac{z_O}{z_m} \frac{z_O - z_m}{1 - z_O} \frac{1}{\sqrt[2]{z_O(z_O - z_m)(z_3 - z_O)}} \times$$
$$F_D\left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_O(1-z_m)}{z_m(1-z_O)}, \frac{z_O}{z_m}, \frac{z_O(z_m - z_3)}{z_m(z_O - z_{3)}}\right) \tag{60}$$

Lauricella's 4^{th} hypergeometric function of m-variables.

$$F_D(\alpha, \boldsymbol{\beta}, \gamma, \mathbf{z}) = \sum_{n_1, n_2, \ldots, n_m = 0}^{\infty} \frac{(\alpha)_{n_1 + \cdots n_m} (\beta_1)_{n_1} \cdots (\beta_m)_{n_m}}{(\gamma)_{n_1 + \cdots + n_m} (1)_{n_1} \cdots (1)_{n_m}} z_1^{n_1} \cdots z_m^{n_m} \tag{61}$$

(tr1)

where

$$\mathbf{z} = (z_1, \ldots, z_m),$$
$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m). \tag{62}$$

The Pochhammer symbol $(\alpha)_m = (\alpha, m)$ is defined by

$$(\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if} \quad m = 0 \\ \alpha(\alpha+1)\cdots(\alpha+m-1) & \text{if}\, m = 1, 2, 3 \end{cases} \tag{63}$$

With the notations $\mathbf{z}^{\mathbf{n}} := z_1^{n_1}\cdots z_m^{n_m}$, $(\boldsymbol{\beta})_{\mathbf{n}} := (\beta_1)_{n_1}\cdots(\beta_m)_{n_m}$, $\mathbf{n}! = n_1!\cdots n_m!$, $|\mathbf{n}| := n_1+\cdots+n_m$ for m-tuples of numbers in (62) and of non-negative integers $\mathbf{n} = (n_1,\cdots n_m)$ the Lauricella series F_D in compact form is:

$$F_D(\alpha,\boldsymbol{\beta},\gamma,\mathbf{z}) := \sum_{\mathbf{n}} \frac{(\alpha)_{|\mathbf{n}|}(\boldsymbol{\beta})_{\mathbf{n}}}{(\gamma)_{|\mathbf{n}|}\mathbf{n}!}\mathbf{z}^{\mathbf{n}} \tag{64}$$

The series admits the following integral representation:

$$F_D(\alpha,\boldsymbol{\beta},\gamma,\mathbf{z}) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-z_1t)^{-\beta_1}\cdots(1-z_mt)^{-\beta_m}\mathrm{d}t \tag{65}$$

which is valid for $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\gamma-\alpha)>0.$. It converges absolutely inside the m-dimensional cuboid:

$$|z_j|<1, (j=1,\ldots,m). \tag{66}$$

For $m=2$, F_D in the notation of Appell becomes the two variable hypergeometric function $F_1(\alpha,\beta,\beta',\gamma,x,y)$.

For purposes of presentation, we define the following tuples of numbers for the beta parameters and the variables of the function F_D that will occur in our closed form solutions:

$$\mathbf{z_j^1}=\left(\frac{z_j-z_m}{1-z_m},\frac{z_m-z_j}{z_m},\frac{z_m-z_j}{z_m-z_3}\right),\;\;\mathbf{j}=1,2,\mathbf{z_1^1}\equiv\mathbf{z_S^1},\mathbf{z_2^1}\equiv\mathbf{z_O^1},$$
$$\mathbf{z_S^1}=\left(\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right),\;\;\mathbf{z_S^2}=\left(\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right),$$
$$\mathbf{z_O^1}=\left(\frac{z_O-z_m}{1-z_m},\frac{z_m-z_O}{z_m},\frac{z_m-z_O}{z_m-z_3}\right),\;\mathbf{z_O^2}=\left(\frac{z_O(1-z_m)}{z_m(1-z_O)},\frac{z_O}{z_m},\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}\right),$$
$$\mathbf{z_S^3}=\left(\frac{z_S(1-z_3)}{z_S-z_3},\frac{z_S}{z_m}\frac{(z_m-z_3)}{(z_S-z_3)},\frac{z_S}{z_S-z_3}\right),$$
$$\boldsymbol{\beta}_\mathbf{3}^\mathbf{1}=\left(2,\frac{1}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{3}^\mathbf{2}=\left(1,\frac{3}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{3}^\mathbf{3}=\left(1,\frac{1}{2},\frac{3}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{3}^\mathbf{4}=\left(1,\frac{1}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{3}^\mathbf{5}=\left(2,\frac{-1}{2},\frac{1}{2}\right),$$
$$\boldsymbol{\beta}_\mathbf{3}^\mathbf{6}=\left(1,\frac{-1}{2},\frac{3}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{3}^\mathbf{7}=\left(1,\frac{-1}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{3}^\mathbf{8}=\left(1,\frac{1}{2},-\frac{1}{2}\right),\;\boldsymbol{\beta}_\mathbf{3}^\mathbf{9}=\left(\frac{1}{2},1,\frac{1}{2}\right),$$
$$\boldsymbol{\beta}_\mathbf{4}^\mathbf{10}=\left(-2,2,\frac{1}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{4}^\mathbf{11}=\left(-1,1,\frac{1}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{4}^\mathbf{\Lambda 1}=\left(1,1,\frac{1}{2},\frac{1}{2}\right),$$
$$\boldsymbol{\beta}_\mathbf{4}^\mathbf{\Lambda 2}=\left(1,1,-\frac{3}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_\mathbf{4}^\mathbf{\Lambda 3}=\left(-1,\frac{1}{2},\frac{1}{2},1\right)\tag{67}$$

and the corresponding 2-tuples for the two-variable Appell's first hypergeometric function F_1:

$$\mathbf{z_A^{1S}}=\left(\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right),\;\;\mathbf{z_A^{1O}}=\left(\frac{z_m-z_O}{z_m},\frac{z_m-z_O}{z_m-z_3}\right),\;\;\mathbf{z_{AaO}^{td}}=\left(\frac{z_O}{z_m},\frac{z_O}{z_m}\frac{z_m-z_3}{z_O-z_3}\right),$$
$$\mathbf{z_A^{2S}}=\left(\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right),\;\;\mathbf{z_A^{2O}}=\left(\frac{z_m}{z_m-z_3}\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)},\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}\right),$$
$$\mathbf{z_{AaS}^{td}}=\left(\frac{z_S}{z_m},\frac{z_S}{z_m}\frac{z_m-z_3}{z_S-z_3}\right),\;\;\mathbf{z_{AO_2}^{1/4}}=\left(\frac{z_m}{z_m-z_3}\frac{(1/4)(z_m-z_3)}{z_m((1/4)-z_3)},\frac{(1/4)(z_m-z_3)}{z_m((1/4)-z_3)}\right),$$
$$\mathbf{z_{AO_1}^{1/4}}=\left(\frac{z_m-\frac{1}{4}}{z_m},\frac{z_m-\frac{1}{4}}{z_m-z_3}\right),\;\;\boldsymbol{\beta}_{\boldsymbol{A}}^{\boldsymbol{ra}}=\left(\frac{1}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_{\boldsymbol{A}}^{\boldsymbol{a1}}=\left(\frac{3}{2},\frac{1}{2}\right),\;\;\boldsymbol{\beta}_{\boldsymbol{A}}^{\boldsymbol{a2}}=\left(\frac{1}{2},\frac{3}{2}\right),\;\;\boldsymbol{\beta}_{\boldsymbol{A}}^{\boldsymbol{a3}}=\left(-\frac{1}{2},\frac{1}{2}\right)\tag{68}$$

The angular integrals of the form $\pm\int_{\theta_S}^{\theta_{\min/\max}}$ in equation (4) are calculated in closed-analytic form as follows:

$$\pm\int_{\theta_S}^{\theta_{\min/\max}}\frac{\mathrm{d}\theta}{\sqrt[2]{\Theta}}=\frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)}$$
$$+[1-\mathrm{sign}(\theta_s\circ\theta_{ms})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}}\times$$
$$F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right) \tag{69}$$

7 Closed form solution for the lens equations.

We now present our closed form solution for the Kerr black hole. We begin the presentation of the solution from the angular terms appearing in 22 Kraniotis,CQG 28 (2011) 085021:

Theorem 2

$$A_2(x_i,y_i,x_S,y_S,m)=2(m-1)\times\left[\frac{\Phi}{|a|}\frac{1}{\sqrt[2]{(z_m-z_3)}}\frac{1}{(1-z_m)}\frac{\pi}{2}F_1\left(\frac{1}{2},1,\frac{1}{2},1,\frac{-z_m}{1-z_m},\frac{z_m}{z_m-z_3}\right)\right]$$
$$+\frac{\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\times F_D\left(\frac{1}{2},\boldsymbol{\beta}_{\mathbf{3}}^{\mathbf{4}},\frac{3}{2},\mathbf{z}_{\mathbf{S}}^{\mathbf{1}}\right)$$
$$+[1-\mathrm{sign}(\theta_S\circ\theta_{mS})]\frac{\Phi}{|a|}\frac{z_S}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt[2]{z_S(z_S-z_m)(z_3-z_S)}}\times$$
$$F_D\left(1,\boldsymbol{\beta}_{\mathbf{3}}^{\mathbf{7}},\frac{3}{2},\mathbf{z}_{\mathbf{S}}^{\mathbf{2}}\right)+$$
$$+\frac{\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_O)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\times F_D\left(\frac{1}{2},\boldsymbol{\beta}_{\mathbf{3}}^{\mathbf{4}},\frac{3}{2},\mathbf{z}_{\mathbf{O}}^{\mathbf{1}}\right)$$
$$+[1-\mathrm{sign}(\theta_O\circ\theta_{mO})]\frac{\Phi}{|a|}\frac{z_O}{z_m}\frac{z_O-z_m}{1-z_O}\frac{1}{\sqrt[2]{z_O(z_O-z_m)(z_3-z_O)}}\times$$
$$F_D\left(1,\boldsymbol{\beta}_{\mathbf{3}}^{\mathbf{7}},\frac{3}{2},\mathbf{z}_{\mathbf{O}}^{\mathbf{2}}\right) \tag{70}$$

Theorem 3

$$A_1(x_i,y_i,x_S,y_S,m)=2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right)+$$
$$\frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\boldsymbol{\beta^{ra}_{A}},\frac{3}{2},\mathbf{z^{1S}_{A}}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)}$$
$$+[1-\mathrm{sign}(\theta_S\circ\theta_{mS})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}}\times F_1\left(\frac{1}{2},\boldsymbol{\beta^{ra}_{A}},\frac{3}{2},\mathbf{z^{2S}_{A}}\right)+$$
$$\frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_O)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\boldsymbol{\beta^{ra}_{A}},\frac{3}{2},\mathbf{z^{1O}_{A}}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)}$$
$$+[1-\mathrm{sign}(\theta_O\circ\theta_{mO})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}}}{\sqrt[2]{z_m-z_3}}\times F_1\left(\frac{1}{2},\boldsymbol{\beta^{ra}_{A}},\frac{3}{2},\mathbf{z^{2O}_{A}}\right) \tag{71}$$

8 Closed form solution for the radial integrals.

We now perform the radial integration assuming $\Lambda=0$:(AO)

For an observer and a source located far away from the black hole, the relevant radial integrals can take the form:

$$\int^r \rightarrow -\int_{r_S}^{\alpha}+\int_{\alpha}^{r_O}\simeq 2\int_{\alpha}^{\infty} \tag{72}$$

For instance we meet the radial integral:

$$\int_{\alpha}^{\infty}\frac{aE}{\Delta}[(r^2+a^2)-a\Phi]\frac{dr}{\sqrt[2]{R}} \tag{73}$$

In order to calculate the contribution to the deflection angle from the radial term we need to integrate the above equation from the distance of closest approach (e.g., from the maximum positive root of the quartic) to infinity. We denote the roots of the quartic by $\alpha,\beta,\gamma,\delta:\alpha>\beta>\gamma>\delta$.

It suffices to proceed with the term:

$$\int_{\alpha}^{\infty}\frac{a\frac{2GM}{c^2}r-a^2\Phi}{\Delta\sqrt[2]{R}}\mathrm{d}r \tag{74}$$

In G. V. Kraniotis, Clas.Quantum.Grav. 28 (2011) 085021 the following theorem was proved:

Theorem 4 $2\int_{\alpha}^{\infty}\frac{a\frac{2GM}{c^2}r-a^2\Phi}{\Delta\sqrt[2]{R}}\mathrm{d}r$

$$
\begin{aligned}
&=2\Bigg[\frac{-2A_{+}^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^{+}}F_D\left(\frac{1}{2},\boldsymbol{\beta_3^9},\frac{3}{2},\mathbf{z_{+}^{r}}\right)\\
&+\frac{A_{+}^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^{+}}\left(-\frac{1}{\kappa_{+}^{\prime 2}}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^r}\right)2\right.\\
&\left.+\frac{1}{\kappa_{+}^{\prime 2}}F_D\left(\frac{1}{2},\boldsymbol{\beta_3^9},\frac{3}{2},\mathbf{z_{+}^{r}}\right)2\right)\\
&+\frac{-2A_{-}^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^{-}}F_D\left(\frac{1}{2},\boldsymbol{\beta_3^9},\frac{3}{2},\mathbf{z_{-}^{r}}\right)\\
&+\frac{A_{-}^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^{-}}\left(-\frac{1}{\kappa_{-}^{\prime 2}}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^r}\right)2\right.\\
&\left.+\frac{1}{\kappa_{-}^{\prime 2}}F_D\left(\frac{1}{2},\boldsymbol{\beta_3^9},\frac{3}{2},\mathbf{z_{-}^{r}}\right)2\right)\Bigg]\\
&\equiv R_2(x_i,y_i) \qquad (75)
\end{aligned}
$$

R2 NHA

Theorem 5 *Also:* $\int_{\alpha}^{\infty}\frac{\mathrm{d}r}{\sqrt{R}}=\frac{1}{\sqrt{(\alpha-\gamma)(\alpha-\delta)}}\frac{\Gamma(1/2)}{\Gamma(3/2)}F_1\left(\frac{1}{2},\boldsymbol{\beta}_A^{\mathbf{ra}},\frac{3}{2},\mathbf{z_A^r}\right).$

We exploit further the lens equations:

$$
\begin{aligned}
&R_1(x_i,y_i)-2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right)\\
&+\cdots=\int^{\xi_S}\frac{\mathrm{d}\xi}{\sqrt{4\xi^3-g_2\xi-g_3}}. \qquad (76)
\end{aligned}
$$

Inverting yields the Weierstraß modular function:

$$\xi_S=\wp(\mathrm{lhs}(76)+\epsilon). \qquad (77)$$

9 Summary of the solution for the lens equation

Let us summarize our closed form solution for the lens equations in Kerr black hole Kraniotis,CQG 28 (2011) 085021:

$$\xi_S = \wp\left(R_1(x_i,y_i) - 2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right) + \cdots + \epsilon\right) \tag{78}$$

The balance equation :

$$\begin{aligned} R_1(x_i,y_i) \equiv 2\int_\alpha^\infty \frac{1}{\sqrt{R}}\mathrm{d}r = A_1(x_i,y_i,x_S,y_S,m) \Leftrightarrow \frac{2}{\sqrt{(\alpha-\gamma)(\alpha-\delta)}}\frac{\Gamma(1/2)}{\Gamma(3/2)}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^r}\right) \\ = 2(m-1)\frac{1}{2|a|}\sqrt{\frac{z_m}{z_m(z_m-z_3)}}\pi F\left(\frac{1}{2},\frac{1}{2},1,\frac{z_m}{z_m-z_3}\right) \\ + \frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^{1S}}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ + [1-\mathrm{sign}(\theta_S\circ\theta_{mS})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}} \times F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^{2S}}\right) \\ + \frac{1}{2|a|}\frac{\sqrt[2]{(z_m-z_O)}}{\sqrt[2]{z_m(z_m-z_3)}}F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^{1O}}\right)\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ + [1-\mathrm{sign}(\theta_O\circ\theta_{mO})]\frac{1}{|a|}\frac{\sqrt[2]{\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}}}{\sqrt[2]{z_m-z_3}} \times F_1\left(\frac{1}{2},\boldsymbol{\beta_A^{ra}},\frac{3}{2},\mathbf{z_A^{2O}}\right), \end{aligned} \tag{79}$$

$$-\phi_S = R_2(x_i,y_i) + A_2(x_i,y_i,x_S,y_S,m) \tag{80}$$

theor1 retsol while the Weierstraß invariants are given in terms of the initial conditions by:

$$g_2 = \frac{1}{12}(\alpha+\beta)^2 - \mathcal{Q}\frac{\alpha}{4}, g_3 = \frac{1}{216}(\alpha+\beta)^3 - \mathcal{Q}\frac{\alpha^2}{48} - \mathcal{Q}\frac{\alpha\beta}{48}. \tag{81}$$

Also $\alpha := -a^2, \beta := \mathcal{Q}+\Phi^2$, $z_S = -\frac{\xi_S + \frac{\alpha+\beta}{12}}{-\alpha/4}$, and ϵ is a constant of integration.

10 Specific examples.

Positions of Images , source for an equatorial observer.

In this case ($\theta_O = \pi/2$), and the equations relating the first integrals of motion to the coordinates on the observer's image plane become:

$$\begin{aligned}\Phi &\simeq -\alpha_i \sin\theta_O = -\alpha_i \\ \mathcal{Q} &\simeq \beta_i^2 + (\alpha_i^2 - a^2)\cos^2\theta_O = \beta_i^2.\end{aligned} \tag{82}$$

Thus the length of the vector on the observer's image plane equals:

$$\sqrt{\alpha_i^2 + \beta_i^2} = \sqrt{\Phi^2 + \mathcal{Q}}. \tag{83}$$

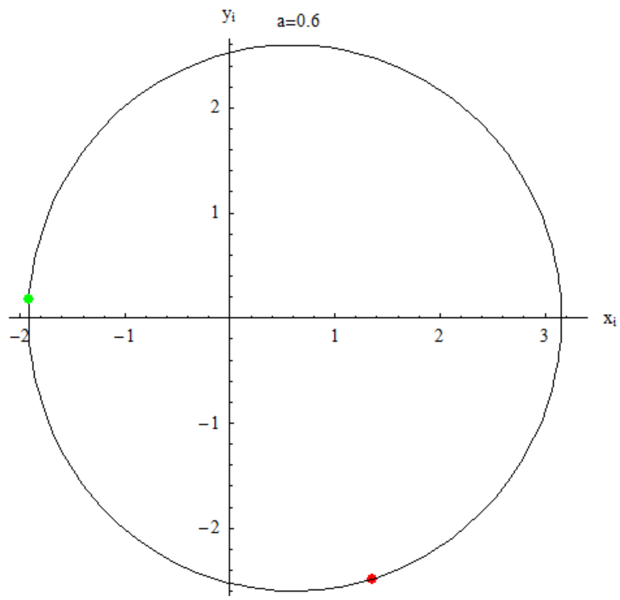
For a choice of initial conditions $a, \Phi, \mathcal{Q}$ we determine values for the observer's image plane coordinates α_i, β_i. Subsequently, we determine the value of z_S that solves the balance equation, exploiting the exact solution for z_S in terms of the Weierstraß elliptic function $\wp$. We must determine at which regions of the *fundamental period parallelogram* the Weierstraß function takes real and negative values. Indeed:

$$\wp(\frac{\omega}{l} + \omega'; g_2, g_3) \in \mathbb{R}^-, for\ x = \frac{\omega}{l} + \omega', \quad l \in \mathbb{R} \tag{84}$$

Having determined θ_S we determine the azimuthal position of the source ϕ_S by (80)

	$a = 0.6, \mathcal{Q} = 24.64563, \Phi = -2.71910$	$a = 0.6, \mathcal{Q} = 0.128, \Phi = 3.839$
$\alpha_i\ (\frac{GM}{c^2})$	2.719110	-3.839
$\beta_i\ (\frac{GM}{c^2})$	-4.9644365239	0.357770876399
$x_i\left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	1.359555	-1.9195
$y_i(\frac{2}{r_O}\frac{GM}{c^2})$	-2.48221826	0.178885
m	3	3
z_S	0.3161007914992452	0.0026145818604
θ_S	55.79°	87.069°
$\Delta\phi$(rad)	-11.086	7.09441
ϕ_S	95.1794°	133.52°
ω	0.5545341990201503500	0.824718843878947
ω'	1.3278669366032567973i	2.9400828459149726i

Assuming that the galactic centre region SgrA*, is a Kerr black hole with mass: $M_{BH} = 4.06 \times 10^6\ M_\odot$ and a distance from the observer to the galactic centre: $r_O = 8$Kpc, the second solution (green image) will require an angular resolution of 19.3102μarcs. This is in the

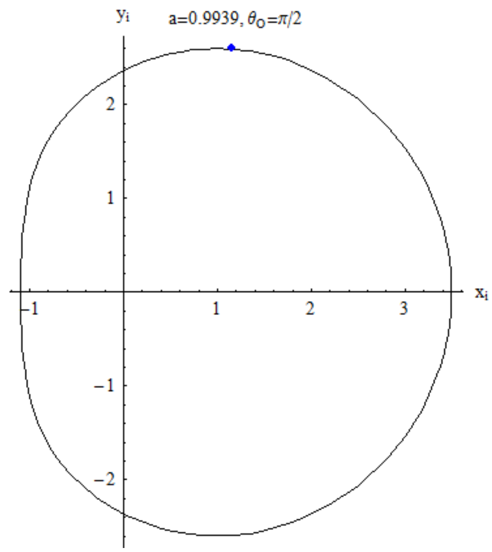


range of experimental accuracy for both the under construction TMT and GRAVITY experiments.

We repeat the analysis for a higher value for the spin of the black hole.

	$a = 0.9939,\ \mathcal{Q} = 27.0220588123,\ \Phi = -2.29885534$
$\alpha_i\ (\frac{GM}{c^2})$ $\beta_i\ (\frac{GM}{c^2})$ $x_i\left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	2.29885534 5.198274599547431 1.14942767
$y_i(\frac{2}{r_O}\frac{GM}{c^2})$	2.5991372997737154
m	3
z_S	0.01378435185109
θ_S	83.2575°
$\Delta\phi$(rad)	−11.243
ϕ_S	104.177°
ω	0.5505433970950226
ω'	1.1288708298860726 i

We observe in this case, that the boundary of the shadow of the black hole is not perfectly circular as it tends to be for low value (in fact for $a = 0$) of the Kerr black hole. The observation of this shadow would be direct evidence of an **event horizon** and the departure from a perfect circle for the ring of light will provide experiment evidence for the spin of



the black hole. This will lead to a measurement of black hole's angular momentum.

11 Magnifications for an equatorial observer in a Kerr black hole.

In this case $(\theta_O = \pi/2)$, equations (15),(16), become:

$$\Phi \simeq -\alpha_i \sin\theta_O = -\alpha_i \tag{85}$$
$$\mathcal{Q} \simeq \beta_i^2 + (\alpha_i^2 - a^2)\cos^2\theta_O = \beta_i^2 \tag{86}$$

and

$$x_S := \frac{\alpha_S}{r_O} = \frac{r_S \sin\theta_S \sin\phi_S}{r_O - r_S \sin\theta_S \cos\phi_S} \tag{87}$$

$$y_S := \frac{\beta_S}{r_O} = \frac{-r_S \cos\theta_S}{r_O - r_S \sin\theta_S \cos\phi_S} \tag{88}$$

$$
\begin{aligned}
\frac{\partial(59)}{\partial x_S} &= \frac{\partial(59)}{\partial z_S}\frac{\partial z_S}{\partial x_S}, \\
\frac{\partial(59)}{\partial z_S} &= \frac{\Phi}{2|a|}\frac{1}{z_m}\frac{1}{(1-z_m)}\frac{1}{\sqrt{z_m-z_3}}\left(\frac{z_m-z_S}{z_m}\right)^{-1/2}\times \\
& F_D\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)+ \\
& \left(\frac{-\Phi}{2|a|}\sqrt[2]{\frac{(z_m-z_S)}{z_m}}\frac{1}{\sqrt[2]{z_m-z_3}}\frac{2}{(1-z_m)}\right)\times\Bigg\{ \\
& F_D\left(\frac{3}{2},2,\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{1}{1-z_m}+ \\
& F_D\left(\frac{3}{2},1,\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{-1}{z_m}+ \\
& F_D\left(\frac{3}{2},1,\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_S-z_m}{1-z_m},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\frac{-1}{z_m-z_3}\Bigg\}+ \\
& (1-\mathrm{sign}(\theta_S\circ\theta_{ms})\,(-1)\Bigg[\Big[\frac{1}{z_m}\frac{z_S-z_m}{1-z_S}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}}+ \\
& \frac{z_S}{z_m}\frac{z_3(z_m-3z_Sz_m+2z_S^2)-z_S(z_m(2-4z_S)+z_S(-1+3z_S))}{2(1-z_S)^2z_S(z_3-z_S)\sqrt{z_S(z_S-z_m)(z_3-z_S)}}\Big]\times \\
& F_D\left(1,1,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)+ \\
& \frac{z_S}{z_m}\frac{z_S-z_m}{(1-z_S)}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}}\Bigg\{ \\
& F_D\left(2,2,-\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\frac{1-z_m}{z_m(1-z_S)^2}+ \\
& F_D\left(2,1,\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\frac{1}{z_m}+ \\
& F_D\left(2,1,\frac{-1}{2},\frac{3}{2},\frac{5}{2},\frac{z_S(1-z_m)}{z_m(1-z_S)},\frac{z_S}{z_m},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{-z_3(z_m-z_3)}{z_m(z_S-z_3)^2}\right)\Bigg\}\Bigg]
\end{aligned}
\tag{89}
$$

Now we calculate the term: $\frac{\partial(??)}{\partial z_S}$. Indeed, calculating the derivatives w.r.t. z_S we derive the expression:

$$\frac{\partial(\mathbf{??})}{\partial z_S}=\frac{1}{2|a|}\frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)}\left(-\frac{1}{2\sqrt{z_m(z_m-z_3)}\sqrt{z_m-z_S}}\right)F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)+$$
$$\frac{1}{2|a|}\frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)}\sqrt{\frac{(z_m-z_S)}{z_m(z_m-z_3)}}\times\left[F_1\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\left(\frac{-1}{z_m}\right)+\right.$$
$$\left.F_1\left(\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_m-z_S}{z_m},\frac{z_m-z_S}{z_m-z_3}\right)\left(\frac{-1}{z_m-z_3}\right)\right]+$$
$$[1-\operatorname{sign}(\theta_S\circ\theta_{ms})]\left[\frac{1}{2|a|}\left(\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)^{-\frac{1}{2}}\left\{\frac{(-z_3)\sqrt{z_m-z_3}}{z_m(z_S-z_3)^2}\right\}\times\right.$$
$$F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)+$$
$$\frac{1}{|a|}\frac{\sqrt{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt{z_m-z_3}}\times\left[F_1\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{5}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{-z_3}{(z_S-z_3)^2}\right)+\right.$$
$$\left.\left.F_1\left(\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{z_m}{z_m-z_3}\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)},\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right)\left(\frac{(-z_3)(z_m-z_3)}{z_m(z_S-z_3)^2}\right)\right]\right] \tag{90}$$

$$\alpha_1=\frac{\partial A_1}{\partial x_S}=(90)\times\frac{\partial z_S}{\partial x_S}=(90)\times\left(-2\cos\theta_S\sin\theta_S\times\frac{\frac{r_S^2\sin\theta_S\cos\theta_S\sin\phi_S}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right) \tag{91}$$

$$\alpha_2=\frac{\partial A_1}{\partial y_S}=(90)\times\frac{\partial z_S}{\partial y_S}=(90)\times\left(-2\cos\theta_S\sin\theta_S\times\frac{\frac{-[r_Or_S\sin\theta_S\cos\phi_S-r_S^2\sin^2\theta_S]}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right) \tag{92}$$

While for the α_3, α_4 terms which contrbute to the expression for the magnification, equation (27), m we derive the expressions:

$$\alpha_3=-\frac{\partial\phi_S}{\partial x_S}-\frac{\partial A_2}{\partial x_S}=-\left(-\frac{\frac{(r_Or_S\sin\theta_S-r_S^2\cos\phi_S)}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right)-(89)\times\left(\frac{\frac{r_S^2\sin\theta_S\cos\theta_S\sin\phi_S}{(r_O-r_S\sin\theta_S\cos\phi_S)^2}}{J_1}\right) \tag{93}$$

,

$$\alpha_4 = -\frac{\partial\phi_S}{\partial y_S} - \frac{\partial A_2}{\partial y_S} = -\frac{\frac{r_O r_S \cos\theta_S \sin\phi_S}{(r_O - r_S \sin\theta_S \cos\phi_S)^2}}{J_1} - (89) \times \frac{\frac{-[r_O r_S \sin\theta_S \cos\phi_S - r_S^2 \sin^2\theta_S]}{(r_O - r_S \sin\theta_S \cos\phi_S)^2}}{J_1} \Bigg) \tag{94}$$

where J_1 denotes the Jacobian:

$$J_1 = \frac{\partial(x_S, y_S)}{\partial(\theta_S, \phi_S)} \tag{95}$$

and

$$\begin{aligned}
\frac{\partial\theta_S}{\partial x_S} &= \frac{(r_S^2 \sin\theta_S \cos\theta_S \sin\phi_S)/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial\theta_S}{\partial y_S} &= \frac{-[r_O r_S \sin\theta_S \cos\phi_S - r_S^2 \sin^2\theta_S]/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial\phi_S}{\partial x_S} &= -\frac{(r_O r_S \sin\theta_S - r_S^2 \cos\phi_S)/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1} \\
\frac{\partial\phi_S}{\partial y_S} &= \frac{r_O r_S \cos\theta_S \sin\phi_S/((r_O - r_S \sin\theta_S \cos\phi_S)^2)}{J_1}
\end{aligned} \tag{96}$$

In producing the results exhibited in eqns (89),(90) in our calculations for the magnification factors we made use of the important identity of Appell's hypergeometric function F_1 and its corresponding generalization for the Lauricella hypergeometric function F_D :

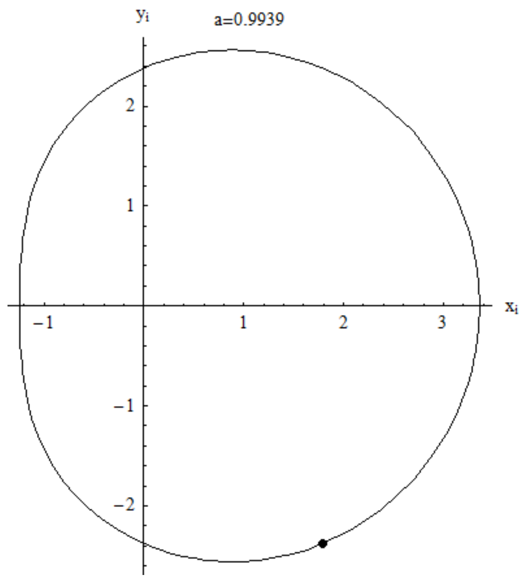
$$\begin{aligned}
\frac{\partial^{m+n} F_1(\alpha, \beta, \beta', \gamma, x, y)}{\partial x^m \partial x^n} &= \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)} \times \\
&\quad F_1(\alpha + m + n, \beta + m, \beta' + n, \gamma + m + n, x, y)
\end{aligned} \tag{97}$$

Similar calculations that we do not exhibit in this talk leads to the derivation of the coefficients β_i.

12 Source and image positions for an observer located at $\theta_O = \frac{\pi}{3}$.

In this case:

$$\Phi = -\alpha_i \frac{\sqrt{3}}{2}, \qquad \mathcal{Q} = \beta_i^2 + \left(\frac{4\Phi^2}{3} - a^2 \right) \frac{1}{4}. \tag{98}$$



	$a = 0.9939, \mathcal{Q} = 25.64563, \Phi = -3.11$	$a = 0.52, \mathcal{Q} = 23.64563, \Phi = -2.85$
$\alpha_i \left(\frac{GM}{c^2}\right)$	3.591118674	3.29089
$\beta_i \left(\frac{GM}{c^2}\right)$	−4.7611506980	−4.58320084657
$x_i \left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	1.79556	1.64545
$y_i \left(\frac{2}{r_O}\frac{GM}{c^2}\right)$	−2.38058	−2.2916004
m	3	3
z_S	0.09097820848	0.5980171072414
θ_S	72.4447°	39.3474°
$\Delta\phi(\text{rad})$	−12.0971	−11.8577
ϕ_S	153.112°	139.395°
ω	0.52792338858688228	0.5571026427501503
ω'	1.119903617249492i	1.389041935594241i

13 Exact solution of the angular integrals in the presence of Λ.

There has been a discussion in the literature as to whether or not the cosmological constant contributes to the gravitational lensing. However, the debate has been restricted to the Schwarzschild-de Sitter spacetime Lake (2007), Sereno,Phys.Rev.D 77(2008), Rindler,Phys.Rev.D76(2007). Let us discuss now the more general case of gravitational lensing in the Kerr-de Sitter spacetime.

The generalized solution for the angular integral (58) in the presence of Λ is given by:

$$\pm\int_{\theta_S}^{\theta_{\min/\max}} = \frac{\Xi^2}{2|H|}\frac{z_m - z_S}{(1-\eta z_m)}\frac{1}{\sqrt[2]{z_m(z_m-z_S)(z_m-z_3)}} \times \Bigg\{ + \frac{\Phi}{(1-z_m)} F_D\left(\frac{1}{2}, \boldsymbol{\beta_4^{\Lambda 1}}, \frac{3}{2}, \boldsymbol{z_\Lambda^{a1}}\right)\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} - aF_D\left(\frac{1}{2}, \boldsymbol{\beta_3^{4}}, \frac{3}{2}, \boldsymbol{z_\Lambda^{a2}}\right)\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\Bigg\}$$
$$+(1-\mathrm{sign}(\theta_S \circ \theta_{mS}))\left[\frac{\Xi^2}{|H|}\frac{z_S}{z_m}\frac{z_m - z_S}{1-\eta z_S}\frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}} \times \left\{-aF_D\left(1, \boldsymbol{\beta_3^{7}}, \frac{3}{2}, \boldsymbol{z_\Lambda^{a4}}\right) + \frac{\Phi}{1-z_S}F_D\left(1, \boldsymbol{\beta_4^{\Lambda 2}}, \frac{3}{2}, \boldsymbol{z_\Lambda^{a3}}\right)\right\}\right] \tag{99}$$

where (H):

$\eta := -\frac{a^2\Lambda}{3}, \mu = \frac{z_S}{z_m}\frac{z_m - z_3}{z_S - z_3}, \lambda = \frac{z_S}{z_m}\left(\frac{1-\eta z_m}{1-\eta z_S}\right), \nu = \frac{z_S}{z_m}\left(\frac{1-z_m}{1-z_S}\right)$ and

$$\boldsymbol{z_\Lambda^{a1}} := \left(\frac{\eta(z_S - z_m)}{1-\eta z_m}, \frac{z_S - z_m}{1-z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3}\right),$$
$$\boldsymbol{z_\Lambda^{a2}} := \left(\frac{\eta(z_S - z_m)}{1-\eta z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3}\right),$$
$$\boldsymbol{z_\Lambda^{a3}} := \left(\lambda, \nu, \frac{z_S}{z_m}, \mu\right), \quad \boldsymbol{z_\Lambda^{a4}} := \left(\lambda, \frac{z_S}{z_m}, \mu\right) \tag{100}$$

Also the integrals $\pm\int_{\theta_{\min/\max}}^{\theta_{\max/\min}} = 2\int_0^{z_m}$ contribute the term:

$$2(m-1)\times\Bigg\{\frac{\Xi^2\Phi}{2|H|}\frac{-z_m}{(1-\eta z_m)(1-z_m)}\frac{1}{\sqrt{z_m^2(z_m-z_3)}} \times F_D\left(\frac{1}{2}, \boldsymbol{\beta_4^{\Lambda 1}}, \frac{3}{2}, \boldsymbol{z_{\Lambda 0}^{a1}}\right)2 + \frac{-\Xi^2 a}{2|H|}\frac{z_m}{\sqrt{z_m^2(z_m-z_3)}}\frac{1}{1-\eta z_m} \times F_D\left(\frac{1}{2}, \boldsymbol{\beta_3^{4}}, \frac{3}{2}, \boldsymbol{z_{\Lambda 0}^{a2}}\right)2\Bigg\} \tag{101}$$

where the tuples of numbers $\boldsymbol{z_{\Lambda 0}^{a1}}$, $\boldsymbol{z_{\Lambda 0}^{a2}}$ appearing in the previous equation are defined by setting $z_S = 0$ in the tuples of numbers $\boldsymbol{z_\Lambda^{a1}}$, $\boldsymbol{z_\Lambda^{a2}}$ respectively. Notice that for $\Lambda = 0$ this reduces to equation (57).

14 Closed-form solution for radial integrals in the presence of Λ.

Assume first $\Lambda > 0$. We need to calculate radial integrals of the form:

$$\int \frac{a\Xi^2}{\Delta_r}((r^2+a^2)-a\Phi))\frac{\mathrm{d}r}{\sqrt[2]{R}} \tag{102}$$

We use the technique of partial fractions from integral calculus:

$$\frac{a\Xi^2}{\Delta_r}((r^2+a^2)-a\Phi)) = \frac{A^1}{r-r_\Lambda^+}+\frac{A^2}{r-r_\Lambda^-}+\frac{A^3}{r-r_+}+\frac{A^4}{r-r_-} \tag{103}$$

where $r_\Lambda^+, r_\Lambda^-, r_+, r_-$ are the four real roots of Δ_r (Der).

For instance, for $r_O, r_S < r_\Lambda^+$ one of the integrals we need to calculate is:

$$\frac{1}{\sqrt{\frac{1}{3}(\mathcal{Q}\Lambda+3\Xi^2(1+\frac{\Lambda}{3}(a-\Phi)^2)}}\int_\alpha^{r_\Lambda^+/2}\frac{A^1\mathrm{d}r}{(r-r_\Lambda^+)\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}} \tag{104}$$

Indeed, we compute in closed form

$$\int_\alpha^{r_\Lambda^+/2}\frac{A^1\mathrm{d}r}{(r-r_\Lambda^+)\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}} = \frac{\rho_1}{\sqrt{\rho_1}}H_\Lambda^+ \times F_D\left(\frac{1}{2},\boldsymbol{\beta}_{\mathbf{4}}^{\mathbf{\Lambda 3}},\frac{3}{2},\boldsymbol{z}_{\mathbf{\Lambda^+}}^{\boldsymbol{r}}\right)\frac{\Gamma(1/2)}{\Gamma(3/2)} \tag{105}$$

where

$$\begin{aligned}
\rho_1 &:= \frac{r_\Lambda^+-\beta}{r_\Lambda^+-\alpha}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta}, \\
\boldsymbol{z}_{\mathbf{\Lambda^+}}^{\boldsymbol{r}} &:= \left(\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\frac{\beta-\gamma}{\alpha-\gamma}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\frac{\beta-\delta}{\alpha-\delta}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\frac{r_\Lambda^+-\beta}{r_\Lambda^+-\alpha}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta}\right) \\
H_\Lambda^+ &:= \frac{\alpha-\beta}{|\beta-\alpha|}\frac{1}{r_{\Lambda^+}-\beta}\frac{1}{\sqrt{\omega(\gamma-\alpha)(\delta-\alpha)}}
\end{aligned} \tag{106}$$

Also the radial integral involved in the lhs in the 'balance' lens equation is computed exactly in terms of the hypergeometric function of Appell F_1:

$$\frac{1}{\sqrt{\frac{1}{3}(\mathcal{Q}\Lambda+3\Xi^2(1+\frac{\Lambda}{3}(a-\Phi)^2)}}\int_{\alpha}^{r_\Lambda^+/2}\frac{\mathrm{d}r}{\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}}$$
$$=\frac{\rho_1}{\sqrt{\mathcal{E}}}\frac{1}{\sqrt{\omega(\gamma-\alpha)(\delta-\alpha)}}\frac{\Gamma(1/2)}{\Gamma(3/2)}F_1\left(\frac{1}{2},\boldsymbol{\beta}^{\boldsymbol{ra}}_{\boldsymbol{A}},\frac{3}{2},\boldsymbol{z}^{\boldsymbol{r}}_{\boldsymbol{A\Lambda+}}\right) \tag{107}$$

where $\mathcal{E}:=\frac{1}{3}(\mathcal{Q}\Lambda+3\Xi^2(1+\frac{\Lambda}{3}(a-\Phi)^2)$, $\omega:=\frac{r_\Lambda^+-\alpha}{r_\Lambda^+-\beta}$, $\boldsymbol{z}^{\boldsymbol{r}}_{\boldsymbol{A\Lambda+}}=\left(\frac{\beta-\gamma}{\alpha-\gamma}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta},\frac{\beta-\delta}{\alpha-\delta}\frac{r_\Lambda^+-2\alpha}{r_\Lambda^+-2\beta}\right)$ and $\alpha,\beta,\gamma,\delta$ denote the roots of the quartic polynomial R in the presence of Λ eqn(6).

Likewise, the generalization of equation (81) is given by:

$$\xi_S=\wp(2\times(107+\cdots+\epsilon) \tag{108}$$

where the Weierstraß invariants take the form

$$\begin{aligned}g_2&=\frac{1}{12}(\alpha_\Lambda+\beta_\Lambda)^2-\mathcal{Q}\frac{\alpha_\Lambda}{4},\\ g_3&=\frac{1}{216}(\alpha_\Lambda+\beta_\Lambda)^3-\mathcal{Q}\frac{\alpha_\Lambda^2}{48}-\mathcal{Q}\frac{\alpha_\Lambda\beta_\Lambda}{48}\end{aligned} \tag{109}$$

and

$$\alpha_\Lambda:=-H^2,\quad \beta_\Lambda:=\mathcal{Q}+\Phi^2\Xi^2. \tag{110}$$

A complete phenomenological analysis of our exact solutions in the presence of the cosmological constant Λwill be a subject of a separate publication. Nevertheless, it is evident from the closed form solutions we derived in this work that the cosmological constant does contribute to the gravitational bending of light.

15 The Polarization vector.

The polarization four-vector f^i of a linearly polarized light ray, which is orthogonal to the direction of propagation, must be parallel transported along a null geodesic with its tangent vector u^i (Penrose). The Kerr geometry is of Petrov type D, and the complex quantity:

$$K_{WP}=(A+iB)(r-ia\cos\theta) \tag{111}$$

is conserved along a null geodesic (Walker, Penrose), where

$$\begin{aligned}A&=(u^tf^r-u^rf^t)+a\sin^2\theta(u^rf^\phi-u^\phi f^r),\\ B&=(r^2+a^2)\sin\theta(u^\phi f^\theta-u^\theta f^\phi)-a\sin\theta(u^tf^\theta-u^\theta f^t).\end{aligned} \tag{112}$$

We can set $f^t = 0$, and using the orthogonality condition $u^i f_i = 0$, to eliminate f^r we obtain the matrix R transforming the θ, ϕ components of the polarization vector at the source into the ones at the observer as follows:

$$\begin{bmatrix} \hat{f}^\theta \\ \hat{f}^\phi \end{bmatrix}_O = R \begin{bmatrix} \hat{f}^\theta \\ \hat{f}^\phi \end{bmatrix}_S , \tag{113}$$

where

$$R = (1+x^2)^{-1/2} \begin{bmatrix} 1 & -x \\ -x & -1 \end{bmatrix}. \tag{114}$$

The parameter x

$$x := \frac{\beta_S \gamma_O + \gamma_S \beta_O}{\gamma_S \gamma_O - \beta_S \beta_O} \tag{115}$$

with

$$\gamma = \frac{\Phi}{\sin\theta} - a \sin\theta \tag{116}$$

and it is valid

$$\beta_S^2 + \gamma_S^2 = \beta_O^2 + \gamma_O^2 = \mathcal{Q} + (\Phi - a)^2. \tag{117}$$

For a geodesic joining the source and observer the parameter x can be computed. For instance for the first example we presented $x = 3.28882$.WORK IN PROGRESS.

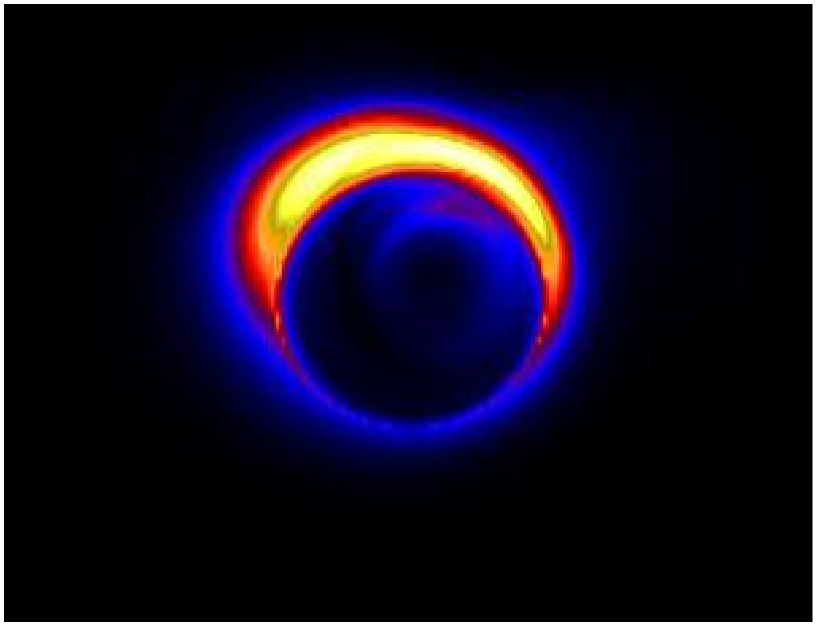
16 Conclusions.

- The precise analytic treatment of Kerr and Kerr-de Sitter black holes as gravitational lenses has been achieved. Full analytic strong-field calculation for the magnification factors was performed.

- Λ does contribute to the gravitational bending of light.

- Important application to the Sgr A* galactic centre black hole (Ghez). Closed form solutions for the periastron advance, frame-dragging and orbital periods for the observed orbits of S-stars in conjuction with the dedicate measurements already constrain significantly the mass of the galactic black hole, and they will eventually determine with high-precision the Sgr A* mass, as well as the distance to the galactic centre. Relativistic observables also depend sensitively on the spin of the black hole. Thus, gravitational lensing in conjuction with the observation of the relativistic effects from timelike geodesy provides a complementary and full test of General Relativity at the strong field regime.

- Fruitfull synergy of various fields of Science: general relativity, astrophysics, cosmology, pure mathematics(Pi1),(Pi2).



- If, the now under development, near-infrared intereferometric GRAVITY experiment (leader R. Genzel for the German team) and the proposed and approved thirty metre telescope (TMT) with leader A. Ghez for the US team, reach the aimed accurary of 10 μarcs and in combination with Very Long Baseline Interferometry (VLBI) observations, then it may be possible for these experiments to detect the effects of strong light bending (the shadow) by the galactic centre black hole. The observation of this shadow would be direct evidence of an event horizon. The departure from complete circularity it will tell us that Sagittarious A* is a spinning black hole.

END, Thank you for your time and attention.

The proof of the shadow of the black hole.

17 Spacelike geodesics.

Such geodesics are traversed by hypothetical faster than light particles and have interesting mathematical properties.

Some additional physical motivation comes from recent progress that has been achieved in deriving global existence theorems for the field equations of General Relativity. The focus of these investigations has shifted to spacelike infinity. This is a regime of ideal points which roughly speaking are endpoints of spacelike geodesics (Friedrich) . On the mathematical side some of the basic theory of conjugate and focal points on spacelike geodesics has emerged, which makes particularly attractive the investigation of obtaining analytic closed form solutions of spacelike geodesics in Kerr black hole spacetimes. Our paper makes the first steps in this direction.

The plausible existence of a superluminal particle in the framework of special relativity (SR) has been discussed by various authors (Bilaniuk et al)(see also Recami) . The energy of a tachyon particle is given by (OHANIAN):

$$E = \frac{|m|c^2}{\sqrt{\frac{\upsilon^2}{c^2}-1}}, \text{ for } \upsilon > c \tag{118}$$

where $|m|$ denotes the magnitude [2] of the tachyon's imaginary rest mass: $m = i|m|$. The energy momentum vector is now

$$p^{\mu} = \left(\frac{|m|c}{\sqrt{\frac{\upsilon^2}{c^2}-1}}, \frac{|m|}{\sqrt{\frac{\upsilon^2}{c^2}-1}}\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{|m|}{\sqrt{\frac{\upsilon^2}{c^2}-1}}\frac{\mathrm{d}y}{\mathrm{d}t}, \frac{|m|}{\sqrt{\frac{\upsilon^2}{c^2}-1}}\frac{\mathrm{d}z}{\mathrm{d}t}\right), \text{ for } \upsilon > c \tag{119}$$

and the dispersion relation is valid

$$p^{\mu}p_{\mu} = (E/c)^2 - p_x^2 - p_y^2 - p_z^2 = -|m|^2c^2. \tag{120}$$

Assuming the gravitational mass of the tachyon is exactly equal to the magnitude of its imaginary rest mass the tachyon moves along spacelike geodesics in a gravitational field. Thus since spacelike geodesics are part of the theory of General Relativity (GTR), tachyons can be considered as implicit ingredients of the theory if the above assumption is valid. We mention at this point earlier works on this matter. Sum (1974) has performed a weak field calculation of the deflection angle of a neutral tachyon in the gravitational field of the Sun assuming a Schwarzschild spacetime geometry. Tachyons in uniform relativistic cosmology have

[2] Sometimes refer to as the *metamass* [**?**].

been considered in Schwarz 2011 and an initial study of the conditions for tachyons to be captured by a Kerr black hole has been performed in Narlikar. However, an exact analytic calculation of the deflection angle for an equatorial orbit in the Kerr gravitational field had not been performed till our calculation.

18 Spacelike geodesics in Kerr-de Sitter spacetime.

Choosing a real affine parameter λ for the spacelike geodesic by $\mathrm{d}\lambda^2 = -\mathrm{d}s^2$, we have the geodesic equation in the usual notation as

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} = 0, \tag{121}$$

where x^i denote the BL coordinates and Γ^i_{jk} the Christoffel symbols of the second kind.

The tachyon motion, as we mentioned earlier, is described by the spacelike geodesics (which are the first integrals of (121)) which take the form[3]:

$$\begin{aligned}
\int \frac{\mathrm{d}r}{\sqrt{R}} &= \int \frac{\mathrm{d}\theta}{\sqrt{\Theta}}, \\
\rho^2 \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} &= -\frac{\Xi^2}{\Delta_\theta \sin^2\theta}(aE\sin^2\theta - L) + \frac{a\Xi^2}{\Delta_r}[(r^2+a^2)E - aL] \\
c\rho^2 \frac{\mathrm{d}t}{\mathrm{d}\lambda} &= \frac{\Xi^2(r^2+a^2)[(r^2+a^2)E - aL]}{\Delta_r} - \frac{a\Xi^2(aE\sin^2\theta - L)}{\Delta_\theta} \\
\rho^2 \frac{\mathrm{d}r}{\mathrm{d}\lambda} &= \pm\sqrt{R} \\
\rho^2 \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} &= \pm\sqrt{\Theta}
\end{aligned} \tag{122}$$

where

$$\begin{aligned}
R &:= \Xi^2[(r^2+a^2)E - aL]^2 - \Delta_r(-r^2 + Q + \Xi^2(L - aE)^2) \\
\Theta &:= [Q + (L-aE)^2\Xi^2 + a^2\cos^2\theta]\Delta_\theta - \Xi^2\frac{(aE\sin^2\theta - L)^2}{\sin^2\theta}
\end{aligned} \tag{123}$$

The constants of motion E, L are associated with the isometries[4] of the Kerr metric while Q denotes Carter's constant, the fourth constant of integration. The spacelike Kerr geodesics are obtained by setting $\Lambda = 0$ in Eqs.(122)-(123).

[3] The tachyon metamass $|m|$, which is one of the first integrals of (121), was set equal to unity without loss of generality.

[4] i.e. they are related to the energy and angular momentum per unit metamass of the tachyon at infinity.

19 Exact solution of equatorial spacelike geodesics in Kerr spacetime

We now proceed to determine the exact solution for the deflection of a neutral tachyon in an equatorial spacelike orbit in Kerr spacetime assuming $\Lambda = 0$. . We have $r^2(\dot{r}) = \sqrt{R}$.This can be rewritten as [5]

$$\dot{r}^2 = E^2 + \frac{a^2 E^2}{r^2} - \frac{L^2}{r^2} + \frac{2GM}{c^2 r^3}(L - aE)^2 + \frac{\Delta}{r^2}. \tag{124}$$

where Δ is obtained by setting $\Lambda = 0$ in equation (**??**) for Δ_r. By defining a new variable $u := 1/r$ we obtain the following expression:

$$u^{-4}\dot{u}^2 = E^2 + a^2E^2u^2 - L^2u^2 + \frac{2GM}{c^2}u^3(L - aE)^2 + (1 + a^2u^2 - \frac{2GM}{c^2}u) \equiv B_{tac}(u). \tag{125}$$

Similarly the geodesic for the azimuth coordinate is given

$$\dot{\phi}^2 = u^4 \frac{A^2(u)}{D^2(u)} \tag{126}$$

where

$$A(u) := L + \alpha_S u(aE - L), \qquad D(u) := 1 + a^2u^2 - \alpha_S u, \qquad \alpha_S := \frac{2GM}{c^2}. \tag{127}$$

Thus we derive the orbital equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}u} = \frac{A(u)}{D(u)} \frac{1}{\sqrt{B_{tac}(u)}}. \tag{128}$$

We now use the technique of partial fractions from integral calculus in order to calculate the deflection of the neutral tachyon's equatorial orbit in the Kerr gravitational field from equation (128). We write:

$$\frac{A(u)}{D(u)} = \frac{A_+}{u_+ - u} + \frac{A_-}{u_- - u} \tag{129}$$

where $u_+ = \frac{r_+}{a^2}, u_- = \frac{r_-}{a^2}$ and

$$r_\pm = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2} \tag{130}$$

denote the radii of the event and Cauchy horizons respectively for the case of a Kerr black hole. Also the quantities A_+, A_- are given by

$$A_+ = \frac{\frac{L}{a^2} + \frac{\alpha_S}{a^2}(aE - L)u_+}{u_- - u_+} \qquad A_- = \frac{\frac{-L}{a^2} - \frac{\alpha_S}{a^2}(aE - L)u_-}{u_- - u_+} \tag{131}$$

[5] For equatorial geodesics $\theta = \pi/2$, $Q = 0$.

In order to calculate the angle of deflection for the tachyon it is necessary to calculate the integral: $\Delta\phi^{GTR}_{Tachyon} = 2\int_0^{u'_2} \mathrm{d}\phi$.Using the formalism developed in previous sections, for computing hyperelliptic integrals in closed analytic form, in terms of Lauricella's hypergeometric function F_D , we compute:

$$\begin{aligned}\Delta\phi^{GTR}_{Tachyon} &= \frac{2}{\sqrt{u'_1-u'_3}}\frac{1}{\sqrt{\frac{\alpha_S(L-aE)^2}{(\frac{GM}{c^2})^3}}}\left\{\frac{A_+}{\frac{GMr_+}{c^2a^2}-u'_3}\left(F_1\left(\frac{1}{2},\boldsymbol{\beta}_A,1,\mathbf{z}_A^{r_+}\right)\pi\right.\right.\\ &\left.-2\sqrt{\frac{-u'_3}{u'_2-u'_3}}F_D\left(\frac{1}{2},\boldsymbol{\beta}_3^4,\frac{3}{2},\mathbf{z}_D^{r_+}\right)\right)\\ &+\frac{A_-}{\frac{GMr_-}{c^2a^2}-u'_3}\left(F_1\left(\frac{1}{2},\boldsymbol{\beta}_A,1,\mathbf{z}_A^{r_-}\right)\pi\right.\\ &\left.\left.-2\sqrt{\frac{-u'_3}{u'_2-u'_3}}F_D\left(\frac{1}{2},\boldsymbol{\beta}_3^4,\frac{3}{2},\mathbf{z}_D^{r_-}\right)\right)\right\} \qquad (132)\end{aligned}$$

where

$$\boldsymbol{\beta}_A = \left(1,\frac{1}{2}\right), \quad \boldsymbol{\beta}_3^4 = \left(1,\frac{1}{2},\frac{1}{2}\right) \qquad (133)$$

and

$$\mathbf{z}_A^{r_\pm} = \left(\frac{u'_2-u'_3}{\frac{GMr_\pm}{c^2a^2}-u'_3},\frac{u'_2-u'_3}{u'_1-u'_3}\right), \quad \mathbf{z}_D^{r_\pm} = \left(\frac{-u'_3}{\frac{GMr_\pm}{c^2a^2}-u'_3},\frac{-u'_3}{u'_1-u'_3},\frac{-u'_3}{u'_2-u'_3}\right) \qquad (134)$$

We have also defined: $u' = u\frac{GM}{c^2}$. The roots of the cubic in this case are real and organized in the ascending order

$$u'_1 > u'_2 > 0 > u'_3 \qquad (135)$$

The angle of deflection δ of a neutral tachyon equatorial trajectory from the gravitational field of a rotating black hole or a rotating central mass is defined to be the deviation of $\Delta\phi^{GTR}_{Tachyon}$ from the transcendental number π

$$\delta^{Tachyon} := \Delta\phi^{GTR}_{Tachyon} - \pi. \qquad (136)$$

We shall apply our closed form solution, Eq.(132) for the deflection angle of an equatorial neutral tachyon's orbit in the gravitational field of Kerr spacetime in two cases. First, we shall compute the deflection angle $\delta^{Tachyon}$, for a tachyon in an equatorial orbit around a Kerr black hole for various values of the involved physical parameters. The physical parameters are the velocity of the tachyon particle υ (at large distances from the central mass), the parameter $E = \frac{1}{\sqrt{v^2/c^2-1}}$, the parameter L and the spin a (Kerr parameter) of the black hole (central spinning mass). Second, in order to test the weak field limit of our exact solutions we compute the deflection angle of a neutral tachyon in an equatorial orbit *i*) in the gravitational field of the Sun *ii*) in the gravitational field of Earth .

20 Deflection of a neutral tachyon in an equatorial orbit in the Kerr black hole spacetime

In this section, we perform a systematic analysis of the computation for the deflection angle of a neutral tachyon on an equatorial orbit in the Kerr black hole's gravitational field by scanning the full parameter space of the spinning black hole. For this purpose, we shall consider equatorial tachyon orbits with angular momentum parallel or antiparallel to the angular momentum of the rotating Kerr black hole. The angular momentum of the tachyon at infinity has the form: $L = bv'E\frac{GM_{\rm BH}}{c^2} = bv'\frac{1}{\sqrt{v^2/c^2-1}}\frac{GM_{\rm BH}}{c^2}$.

We start our analysis, by computing the deflection angle versus the impact factor b, assuming the angular momentum for the equatorial spacelike orbit is parallel to the spin of the black hole.We present our results in Figure 1 for two different values of the black hole's spin and for $v = (1+2.37\times10^{-6})c$. We observe from the figure the strong dependence of the deflection angle on the spin of the black for small values of the parameter b.

We repeat the analysis assuming now that the angular momentum of the equatorial tachyonic orbit is antiparallel to the spin of the black hole. We exhibit our results in Figure 2.

We now repeat the computations assuming a tachyon speed $v = \sqrt{2}c$. We display our results for the deflection angle vs the parameter b for the cases of parallel and antiparallel angular momenta in Figures 3, 4 respectively. For transcendental tachyons (i.e. FLT particles with very high superuminal speeds) the corresponding results are displayed in Fig. 5 and Fig. 6, choosing $v = 10^6c$.

We wish now to determine the dependence of the deflection angle $\delta^{Tachyon}$ on the superluminal speed. For this purpose, we compute $\delta^{Tachyon}$ as a function of v for fixed value of the parameter b for different values of the black hole's spin. Our results are summarized in Figures 7-8. As it is evident from Fig. 8 our computatations show an interesting

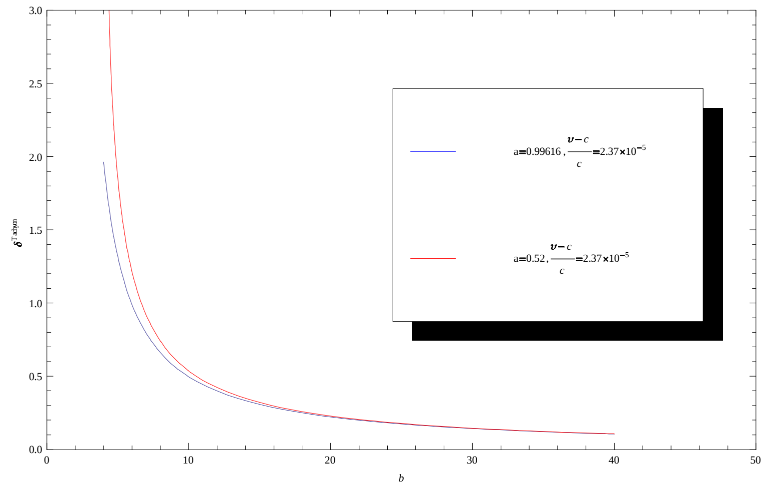


Figure 1: Deflection angle for an equatorial neutral tachyon orbit around a Kerr black hole with angular momentum coalligned to black hole's spin. We present our results for two different values for the magnitude of Kerr black hole's angular momentum. The tachyon's velocity was chosen to be $\upsilon = (1 + 2.37 \times 10^{-5})c$.

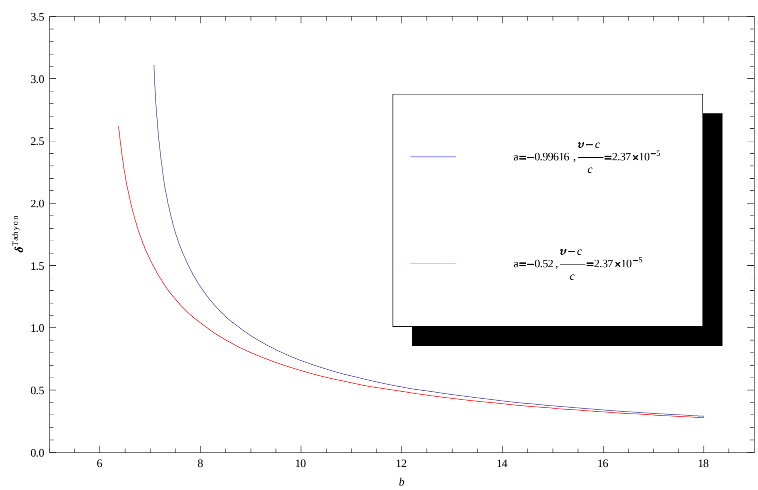


Figure 2: Deflection angle for an equatorial neutral tachyon orbit around a Kerr black hole with angular momentum antiparallel to black hole's spin. We present our results for two different values for the magnitude of Kerr black hole's angular momentum. The tachyon's velocity was chosen to be $\upsilon = (1 + 2.37 \times 10^{-5})c$.

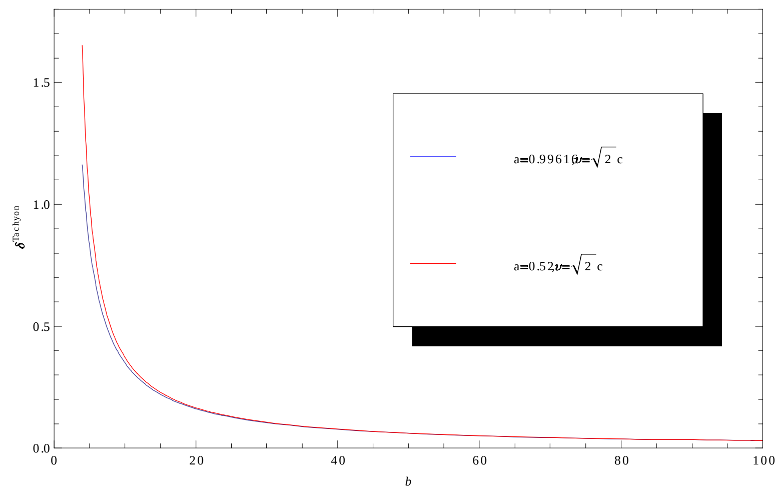


Figure 3: The deflection angle that a neutral tachyon on an equatorial motion in the Kerr's black hole gravitational field undergoes vs the impact parameter assuming that the orbital angular momentum is coalligned to the spin's black hole. We present our results for 2 different values of the angular momentum of Kerr's black hole. We assume $\upsilon = \sqrt{2}c$.

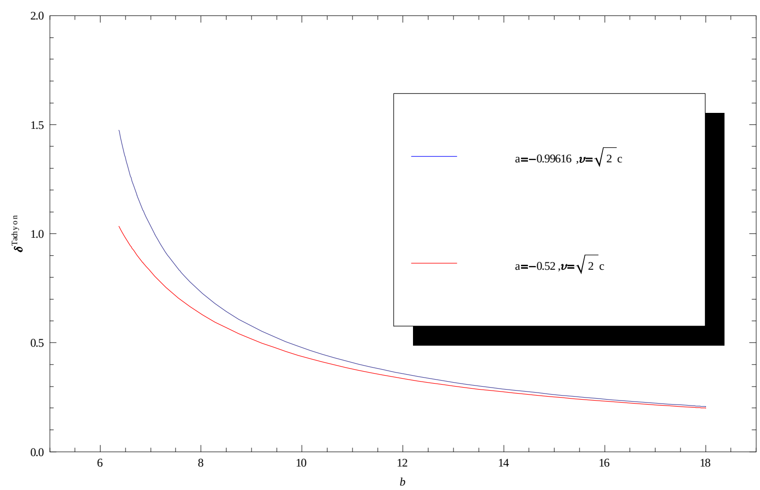


Figure 4: The deflection angle that a neutral tachyon on an equatorial motion in the Kerr's black hole gravitational field undergoes vs the impact parameter assuming that the orbital angular momentum is antiparallel to the spin's black hole. We present our results for 2 different values of the angular momentum's magnitude of Kerr's black hole. We assume $\upsilon = \sqrt{2}c$.

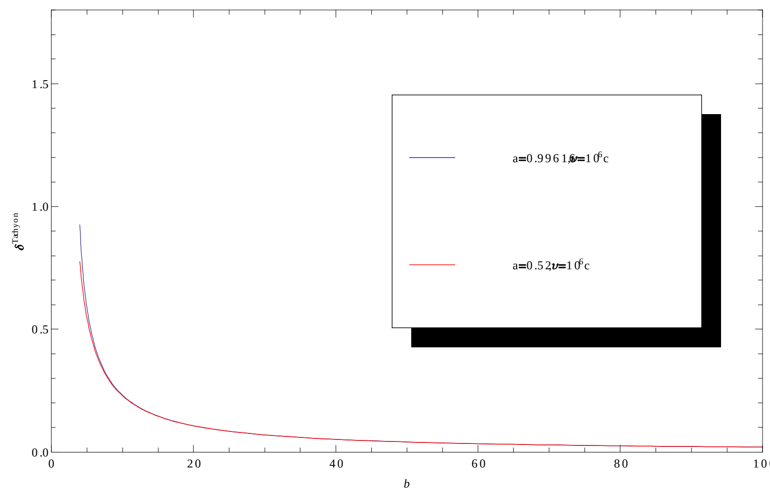


Figure 5: The deflection angle that a neutral tachyon on an equatorial motion in the Kerr's black hole gravitational field undergoes vs the impact parameter assuming that the orbital angular momentum is coalligned to the spin's black hole. We present our results for 2 different values of the angular momentum of Kerr's black hole. We assume $\upsilon = 10^6 c$.

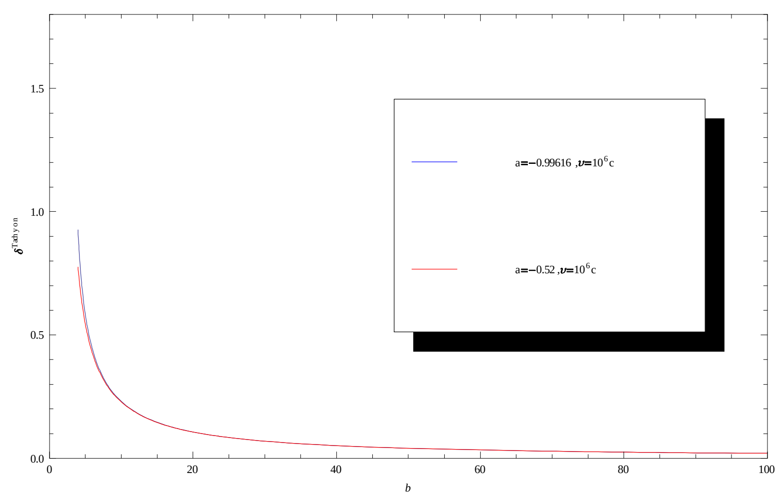


Figure 6: The deflection angle that a neutral tachyon on an equatorial motion in the Kerr's black hole gravitational field undergoes vs the impact parameter assuming that the orbital angular momentum is antiparallel to the spin's black hole. We present our results for 2 different values for the magnitude of the angular momentum of Kerr's black hole. We assume $\upsilon = 10^6 c$.

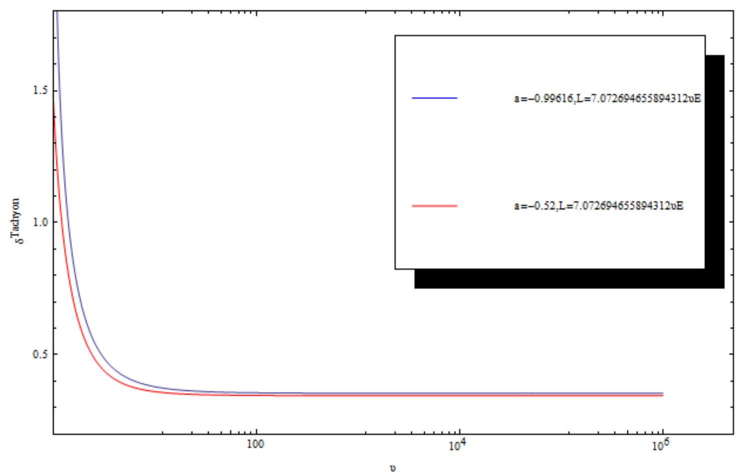


Figure 7: The deflection angle $\delta^{Tachyon}$ of a neutral tachyon versus its superluminal velocity υ, for two different values of the black hole's spin (assumed in opposite direction to tachyon's orbital angular momentum) and fixed coefficient b.

new phenomenon, namely : a *non-monotonic dependence* of the deflection angle on the superluminal speed for high values of the spin of the black hole and for low values for the parameter b. This is a new strong field effect of the Kerr black hole.

21 Deflection of neutral tachyon in an equatorial orbit in the Kerr (Sun's) gravitational field.

Assuming that the gravitational field of the spinning Sun is described by the Kerr spacetime geometry we compute the deflection angle of a tachyon in an unbound equatorial orbit using our closed form solution Eq.(136). We repeat the calculation for three values of the velocity of the tachyon particle. Namely: 1) $\upsilon = (1 + 2.48 \times 10^{-5})c$ (same as OPERA's experimental value) 2) $\upsilon = \sqrt{2}c$ 3) $\upsilon = 10^6 c$. The parameter E is determined by the formula: $E = \frac{1}{\sqrt{\upsilon^2/c^2-1}}$. Taking the point of closest approach $r_0 = R_\odot = 6.9551 \times 10^8$ m [?] the parameter L is chosen to have the value: $L = 471013.2781620\upsilon' E \frac{G_N M_\odot}{c^2}$ where $\upsilon' := \upsilon/c$.Our results are summarized in Table 1. We note that the computed exact value of the deflection angle for case 1, i.e.a case in which a neutral tachyon moves with a velocity equals to the one determined by OPERA experiment (**??**), differs from the deflection angle that light experiences in the Sun's gravitational field at the fifth decimal point.

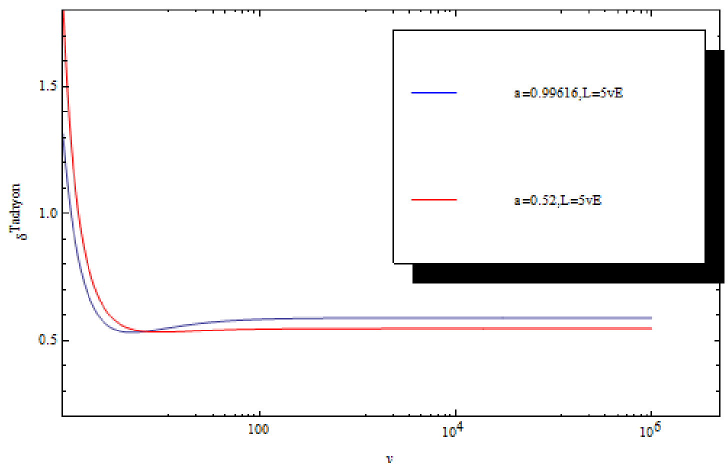


Figure 8: The deflection angle $\delta^{Tachyon}$ versus the particle's superluminal velocity υ,for two different values of the black hole's spin a (assumed co-aligned with tachyon's orbital angular momentum) and fixed parameter $b = 5$.

1: $\upsilon = (1 + 2.48 \times 10^{-5})c$	2: $\upsilon = \sqrt{2}c$	3: $\upsilon = 10^6 c$	$\upsilon^{Phot} = c$
$\delta_{\odot}^{Tach} = 1.75164$arcs	$\delta_{\odot}^{Tach} = 1.31376$arcs	$\delta^{Tach} = 0.875837$arcs	$\delta_{\odot}^{phot} = 1.75168 arcs$

Table 1: Deflection of a neutral tachyon's equatorial trajectory in the gravitational field of the Sun assuming Kerr spacetime geometry. The value of the Kerr parameter was chosen to be $a = 0.2158\frac{GM_{\odot}}{c^2}$. For comparison we also display the value of the deflection angle for light rays $\delta_{\odot}^{photon}$.

Indeed, we calculate using the exact analytic solution obtained in [?] that the deflection angle that a light ray experiences grazing the surface of the Sun is: $\delta_{\odot}^{photon} = 1.75168\,\text{arcsec}$. In fact, by computing the difference $\Delta\delta := \delta_{\odot}^{photon} - \delta_{\odot}^{Tachyon}$ as a function of the dimensionless speed difference $\psi := \frac{\upsilon - c}{c}$, we obtain: $\Delta\delta = 0.0000434399\,\text{arcsec}$ for $\psi = 2.48 \times 10^{-5}$ and $\Delta\delta = 0.0000415136\,\text{arcsec}$ for $\psi = 2.37 \times 10^{-5}$.

For transcendental tachyon velocities, i.e. $\upsilon \gg c$,(case 3 in Table 1) we observe that $\delta_{\odot}^{Tachyon} \rightarrow \frac{\delta_{\odot}^{photon}}{2}$ and this agrees with a perturbative (weak-field) calculation performed in [?] for the static Schwarzchild spatime. This is a nice check of the perturbative limit of our exact solutions for the deflection angle of the superluminal particle in the Kerr spacetime.

21.1 Deflection of a neutral tachyon in an equatorial orbit by the gravitational field of Earth

Assuming the Earth's gravitational field is described by a Kerr spacetime, we shall compute the deflection angle of a neutral tachyon in an equatorial orbit in Earth's gravitation. There is a further novelty in this calculation. The Kerr parameter that corresponds to the angular momentum of Earth is equal to $a_{\oplus} = 329.432\,\mathrm{cm} = 371.398(2GM_{\oplus}/c^2)$[?]. Thus the two roots of the polynomial $D(u)$ are complex-conjugate. Therefore, when we calculate exactly the hyperelliptic integral, two of the variables of Lauricella's function F_D are complex-conjugates. This is fine as long as their modules are less than 1,which indeed it is the case in our computations.

Thus

$$\begin{aligned}\int \mathrm{d}\phi &= 2\int_0^{u_2'} \frac{A(u)}{D(u)} \frac{1}{\sqrt{B_{tac}(u)}} \mathrm{d}u \\ &= 2\int_0^{u_2'} \frac{\mathrm{d}u' L}{a^2\left(\frac{GM_{\oplus}r_+}{c^2a^2} - u'\right)\left(\frac{GM_{\oplus}r_-}{c^2a^2} - u'\right)} \frac{1}{\sqrt{\frac{\alpha_S(L-aE)^2}{(GM_{\oplus}/c^2)^3}}} \frac{1}{\sqrt{(u'-u_3')(u_1'-u')(u_2'-u')}} + \\ & 2\int_0^{u_2'} \frac{\alpha_S(aE-L)\mathrm{d}u' u'}{a^2\left(\frac{GM_{\oplus}r_+}{c^2a^2} - u'\right)\left(\frac{GM_{\oplus}r_-}{c^2a^2} - u'\right)\sqrt{\frac{\alpha_S(L-aE)^2}{(GM_{\oplus}/c^2)^3}}\sqrt{(u'-u_3')(u_1'-u')(u_2'-u')}}\end{aligned} \tag{137}$$

Now applying the transformation

$$\boxed{u' = u_2'(1-t),}$$

we have:

$$\frac{GM_{\oplus}r_{\pm}}{c^2a^2} - u' = \left(\frac{GM_{\oplus}r_{\pm}}{c^2a^2} - u_2'\right)\left[1 + \frac{tu_2'}{\frac{GM_{\oplus}r_{\pm}}{c^2a^2} - u_2'}\right], \quad u_2' - u' = u_2't, \tag{138}$$

$$u_1' - u' = (u_1' - u_2')\left[1 + \frac{u_2't}{u_1' - u_2'}\right], \quad u' - u_3' = (u_2' - u_3')\left[1 - \frac{u_2't}{u_2' - u_3'}\right] \tag{139}$$

thus we tranform our integral onto the integral representation of Lauricella's function F_D of four-variables (see also Appendix):

$$\int \mathrm{d}\phi = \Delta\phi^{GTR}_{NT\oplus}$$
$$= \frac{2}{a^2\left(\frac{GM_{\oplus}r_+}{c^2a^2}-u_2'\right)\left(\frac{GM_{\oplus}r_-}{c^2a^2}-u_2'\right)\sqrt{u_2'(u_2'-u_3')(u_1'-u_2')}\sqrt{\frac{\alpha_S(L-aE)^2}{(GM_{\oplus}/c^2)^3}}}\times$$
$$\left[\alpha_S(aE-L)u_2'^2\frac{\Gamma(1/2)\Gamma(2)}{\Gamma(5/2)}F_D\left(\frac{1}{2},\boldsymbol{\beta}^1_4,\frac{5}{2},\mathbf{z}^{\oplus}_r\right)+\right.$$
$$\left. Lu_2'\frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)}F_D\left(\frac{1}{2},\boldsymbol{\beta}^1_4,\frac{3}{2},\mathbf{z}^{\oplus}_r\right)\right] \tag{140}$$

where

$$\boldsymbol{\beta}^1_4=\left(1,1,\frac{1}{2},\frac{1}{2}\right),\quad \mathbf{z}^{\oplus}_r=\left(\frac{-u_2'}{\frac{GM_{\oplus}r_+}{c^2a^2}-u_2'},\frac{-u_2'}{\frac{GM_{\oplus}r_-}{c^2a^2}-u_2'},\frac{u_2'}{u_2'-u_3'},\frac{-u_2'}{u_1'-u_2'}\right) \tag{141}$$

We apply our analytic solution eq.(140) for computing the deflection angle of a neutral tachyon in the gravitational field of Earth. We choose first, as value for the velocity of the tachyon the central value of OPERA experiment $\upsilon=(1+2.48\times10^{-5})c$. Taking the point of closest approach as the radius (equatorial) of Earth $r_0=R_{\oplus}=6.378137\times10^6\,\mathrm{m}$ [?] we have that the parameter $L=\upsilon' E\times1.438127796068894\times10^9$. For the Kerr parameter of Earth we choose the above mentioned value. Then we computed the deflection angle with the result: $\delta^{GTR}_{NT\oplus}=\Delta\phi^{GTR}_{NT\oplus}-\pi=2.78132\times10^{-9}\,\mathrm{rad}=0.000573689\,\mathrm{arcsec}$. We repeated the computations for different tachyon velocities. For $\upsilon=\sqrt{2}c$, we obtained the result $\delta^{GTR}_{NT\oplus}=\Delta\phi^{GTR}_{NT\oplus}-\pi=2.08604\times10^{-9}\mathrm{rad}=0.000430278\ \mathrm{arcsec}$. Finally, for a (transendental) superluminal velocity $\upsilon=10^6c$ we computed the deflection: $\delta^{GTR}_{NT\oplus}=1.390697\times10^{-9}\,\mathrm{rad}=0.000286852\,\mathrm{arcsec}$.

22 Conclusions

In this part, we investigated tachyon orbits (spacelike geodesics) in Kerr spacetime. More specifically, we derived the closed form solution for the deflection angle of a neutral tachyon in an equatorial orbit in the gravitational field of Kerr spacetime. The solution was expressed elegantly in terms of Lauricella's multivariable hypergeometric function F_D.We applied our exact solutions in three cases: 1) we calculated the deflection of a neutral tachyon by the gravitational field of a rotating Kerr black hole, for different values of the velocity of the tachyon and the spin of the black hole. We note the strong dependence of the deflection angle on the spin of the spinning black hole for low tachyon velocities, especially for lower values for the parameter L. We investigated both cases in which first, the spin of the black hole is coalligned to the orbital tachyon's momentum and second when the source's and test particle's angular momentums are antiparallel. Large magnitudes for the deflection angle were produced see table **??**.2) we calculated the deflection of equatorial neutral tachyon orbits by the gravitational field of our Sun assuming a curved Kerr spacetime geometry. For low tachyon velocities the deflection angle was calculated to have a value $\delta_{\odot}^{Tachyon} = 1.75164\,\text{arcsec}$ a value that differs in the fifth decimal figure from the amount of deflection that light experiences by the gravitation field of our Solar system star: $\delta_{\odot}^{Photon} = 1.75168\,\text{arcsec}$. For high tachyon velocities $\upsilon \gg c$ the calculated deflection angle decreases to half the value of δ at low superluminal velocities. 3) we calculated the deflection angle of an equatorial tachyon trajectory in the gravitational field of Earth assuming a Kerr geometry. There is a further novelty in the calculation. The solution for δ is expressed in terms of Lauricella's hypergeometric function F_D of four variables two of which are complex-conjugates. The neutral tachyon undergoes a small deflection of 2.78132×10^{-9} radians$\sim 0.000573689\,\text{arcsec}$, for a superluminal velocity $\upsilon = (1 + 2.48 \times 10^{-5})c$. Thus if tachyons do exist and move on spacelike geodesics they undergo a deflection by the gravitational field of the rotating central mass. The deflection exhibits a strong dependence on the superluminal velocity and the spin of the rotating mass. This gravitational effect is in principle measurable.