Matrix theory compactifications on twisted tori

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A.C. and Larisa Jonke, arXiv:1202.4310 [hep-th]

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Introduction and Motivation

Superstring / M theory: good candidates for unified description of our current knowledge of nature.

Most notably, they contain gravity and promise to reconcile it with quantum theory.

Full of beautiful structures, dualities etc.

Most of our knowledge within perturbative string theory.

Access to non-perturbative regime?

Matrix Models as non-perturbative definitions of string / M theory.

Banks, Fischler, Shenker, Susskind '96, Ishibashi, Kawai, Kitazawa, Tsuchiya '96

Framework to address profound conceptual problems, study brane dynamics, test symmetries-dualities, analytically and numerically.

Particle physics models, connection to low-energy physics.

Aoki '10, A.C., Steinacker, Zoupanos '11

Cosmological implications, expanding universe.

Kim, Nishimura, Tsuchiya '11

Another interesting direction: Compactifications of MM.

Banks, Fischler, Shenker, Susskind '96, Taylor '96

Matrix compactifications on multidimensional tori $\rightsquigarrow$ striking relations to non-commutative geometry.

Connes, Douglas, Schwarz '97

$\rightsquigarrow$ NC deformations in correspondence to supergravity fluxes.

Douglas, Hull '97, Brace, Morariu, Zumino '98, Kawano, Okuyama '98

Q: Matrix compactifications on twisted tori?

Lowe, Nastase, Ramgoolam '03

Twisted tori:

- ✔ supersymmetric backgrounds rich in fluxes.
- ✔ vacua of heterotic, type II, M theory.

Kaloper, Myers '99, Kachru, Schutz, Tripathy, Trivedi '02,

Hull, Reid-Edwards '05,'06, Grana, Minasian, Petrini, Tomasiello '06

One step closer to non-geometric fluxes.

Dabholkar, Hull '02, Flournoy, Wecht, Williams '04,...

Study the BFSS model on multidimensional twisted tori.

Overview

Twisted tori as nilmanifolds

Matrix compactifications on (twisted) tori

Relations between backgrounds

Conclusions and open questions

Twisted tori as nilmanifolds

Nilmanifolds Mal'cev '51

Smooth manifolds $\mathcal{M}$ of the form A/Γ
A: Nilpotent Lie group, Γ: Discrete subgroup of A

Classification of nilpotent Lie algebras $\mathcal{A}_d$:
Morozov '58, Mubarakzyanov '63, Patera et.al. '75

- 1 in 3D
- 1 in 4D
- 6 in 5D
- 22 in 6D

Nilpotency $\rightsquigarrow$ upper triangular matrices...Facilitates construction of $\mathcal{M}$.

Compactness criterion for $\mathcal{M}$: unimodularity of $\mathcal{A}$ ($f^a{}_{ab} = 0$).

3D example

3D nilpotent Lie algebra $\mathcal{A}_3$: $[X_2, X_3] = X_1$.

Upper triangular basis:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Group element: $g = \begin{pmatrix} 1 & x^2 & x^1 \\ 0 & 1 & x^3 \\ 0 & 0 & 1 \end{pmatrix}, x^i \in \mathbb{R}.$

Restriction to Γ: $g|_\Gamma = \begin{pmatrix} 1 & \gamma^2 & \gamma^1 \\ 0 & 1 & \gamma^3 \\ 0 & 0 & 1 \end{pmatrix}, \gamma^i \in \mathbb{Z}.$

Compact nilmanifold: A_3/Γ.

Some geometry

Invariant 1-form: $e = \begin{pmatrix} 0 & dx^2 & dx^1 - x^2 dx^3 \\ 0 & 0 & dx^3 \\ 0 & 0 & 0 \end{pmatrix}.$

Components: $e^1 = dx^1 - x^2 dx^3, \quad e^2 = dx^2, \quad e^3 = dx^3.$

Metric: $ds^2 = \delta_{ab} e^a e^b$.

Dual vector fields: $\tilde{e}_1 = \partial_1, \quad \tilde{e}_2 = \partial_2, \quad \tilde{e}_3 = \partial_3 + x^2 \partial_1.$
$\tilde{e}_1$ generates an isometry...They satisfy: $[\tilde{e}_2, \tilde{e}_3] = \tilde{e}_1$.

Why "twisted tori"?

For a torus T^3, with covering group $\mathbb{R}^3$ and lattice $\mathbb{Z}^3$,

$$(x^1, x^2, x^3) \sim (x^1 + 1, x^2, x^3) \sim (x^1, x^2 + 1, x^3) \sim (x^1, x^2, x^3 + 1),$$

x^i the toroidal coordinates.

For the 3D nilmanifold:

$$(x^1, x^2, x^3) \sim (x^1+1, x^2, x^3) \sim (x^1, x^2, x^3+1) \sim (x^1+x^3, x^2+1, x^3).$$

$\rightsquigarrow$ twisted fibration of a T^2 fiber over an S^1 base.

- ✔ The T^2 geometry changes as it traverses S^1 $\rightsquigarrow$ twisted $\tilde{T}^3$.

$\rightsquigarrow$ Similar in more dimensions...

T-duality approach

Alternatively, square T^3 with N units of NS-NS flux $H = dB$:

- ✔ Metric: $ds^2 = \delta_{ab} dx^a dx^b$.
- ✔ B-field: $B_{31} = N x^2$.

Perform a T-duality along x^1 using the Buscher rules.

In the T-dual frame:

- ✔ Metric: $ds^2 = \delta_{ab} e^a e^b \rightsquigarrow e^a$ of the twisted torus.
- ✔ B-field: $B = 0$.

$\rightsquigarrow$ works in more dimensions too.

Matrix compactifications on (twisted) tori

Matrix Models

Matrix theory: suggested as non-perturbative definition of M-theory.

Banks, Fischler, Shenker, Susskind '96

Action:

$$\mathcal{S}_{BFSS} = \frac{1}{2g}\int dt\left[Tr\left(\dot{\mathcal{X}}_a\dot{\mathcal{X}}_a - \frac{1}{2}[\mathcal{X}_a,\mathcal{X}_b]^2\right) + 2\psi^T\dot{\psi} - 2\psi^T\Gamma^a[\psi,\mathcal{X}_a]\right],$$

$\mathcal{X}_a(t)$: 9 time-dependent $N \times N$ Hermitian matrices,
ψ: fermionic superpartners, Γ^a: rep. of $SO(9)$.

EOM: $\ddot{\mathcal{X}}_a + [\mathcal{X}_b,[\mathcal{X}^b,\mathcal{X}_a]] = 0.$

IKKT: non-perturbative type IIB superstring.

Ishibashi, Kawai, Kitazawa, Tsuchiya '96

10 Hermitian matrices...and action:

$$\mathcal{S}_{IKKT} = \frac{1}{2g}Tr\left(-\frac{1}{2}[\mathcal{X}_a,\mathcal{X}_b]^2 - \bar{\psi}\Gamma^a[\mathcal{X}_a,\psi]\right).$$

Toroidal compactification (no twist) Connes, Douglas, Schwarz '97

Specific restriction of the matrix action...For a T^3 compactification:

$$
\begin{aligned}
\mathcal{X}_1 + R_1 &= U_1 \mathcal{X}_1 U_1^{-1}, \\
\mathcal{X}_2 + R_2 &= U_2 \mathcal{X}_2 U_2^{-1}, \\
\mathcal{X}_3 + R_3 &= U_3 \mathcal{X}_3 U_3^{-1}, \\
\mathcal{X}_a &= U_i \mathcal{X}_a U_i^{-1}, \quad a \neq i, \quad a = 1, \ldots, 9, \quad i = 1, 2, 3.
\end{aligned}
$$

Determine:

- ✔ The form of the Hermitian matrices $\mathcal{X}$ and...
- ✔ ...the set of unitary matrices U

defining a consistent background of the compactified model.

Ordinary T^3

The compactification conditions are solved by:

$$\begin{aligned} \mathcal{X}_i &= iR_i\mathcal{D}_i, \quad \mathcal{X}_m = \mathcal{A}_m, \quad m = 4, \ldots, 9, \\ U_1 &= e^{ix^1}, \quad U_2 = e^{ix^2}, \quad U_3 = e^{ix^3}, \end{aligned}$$

x^i: coordinates of T^3,
$\mathcal{D}_i$: covariant derivatives $\mathcal{D}_i = \partial_i - i\mathcal{A}_i$,
U-algebra: $[U_i, U_j] = 0$.

The bosonic action becomes that of a (1+3)D sYM theory.

- ✔ Directly generalized to (1+d)D for T^d...

Non-commutative T^3_θ

More general configurations allowed...

Define the operator: $Q_{(ij)} = U_i U_j U_i^{-1} U_j^{-1}, \quad i \neq j.$

It satisfies: $[Q_{(ij)}, \mathcal{X}_a] = 0, \quad \forall\, a = 1, \ldots, 9.$

Therefore it is a scalar operator $\rightsquigarrow$ $U_i U_j = \lambda_{ij} U_j U_i, \quad \lambda_{ij} = e^{2\pi i \theta^{ij}}.$

$\theta^{ij} = 0 \rightarrow$ boils down to the previous case on the standard torus.
But $\theta^{ij} \neq 0$ is new and Us are non-commuting operators...

Compactification on non-commutative T^3_θ.

Coordinates $x^i \to$ Operators $\hat{x}^i$ with $[\hat{x}^i, \hat{x}^j] = -2\pi i\theta^{ij}$.

- ✔ $\mathcal{A}$-fields depend on a set of operators $\hat{U}$, commuting with U:

$$\hat{U}_i = e^{i\hat{x}^i - 2\pi\theta^{ij}\hat{\partial}_j},$$

  satisfying dual relations $\hat{U}_i\hat{U}_j = e^{2\pi i\hat{\theta}^{ij}}\hat{U}_j\hat{U}_i, \quad \hat{\theta}^{ij} = -\theta^{ij}$.

  Brace, Morariu, Zumino '98

- ✔ The sYM theory lives on the dual NC torus with deformation parameter $\hat{\theta}$.

Non-commutative T^3_x

Q: Are there alternative ways to solve the compactification conditions?
Can we have a non-costant deformation?

There are solutions with $\hat{x}$-algebra: $[\hat{x}^1, \hat{x}^3] = iN\hat{x}^2$,
and U-algebra: $U_1 U_3 = e^{-iN\hat{x}^2} U_3 U_1$.

- ✔ NC torus T^3_x with non-constant non-commutativity...
- ✔ ...and constant non-associative structure on the phase space:

$$[\hat{\partial}_2, [\hat{x}^1, \hat{x}^3]] + (\text{cyclic permutations}) = iN$$

This should be related to the presence of N units of H-flux...

Non-associativity in H-flux backgrounds also in CFT computations.

Cornalba, Schiappa '01, Blumenhagen, Plauschinn '10, Lüst '10

Blumenhagen, Deser, Lüst, Plauschinn, Rennecke '11

Twisted compactifications

Restrict the action by imposing conditions corresponding to the twisted identifications for nilmanifolds... For the twisted $\tilde{\mathrm{T}}^3$ case:

$$\begin{aligned}
\mathcal{X}_1 + R_1 &= U_1 \mathcal{X}_1 U_1^{-1}, \\
\mathcal{X}_2 + R_2 &= U_2 \mathcal{X}_2 U_2^{-1}, \\
\mathcal{X}_3 + R_3 &= U_3 \mathcal{X}_3 U_3^{-1}, \\
\mathcal{X}_1 + R_2 \mathcal{X}_3 &= U_2 \mathcal{X}_1 U_2^{-1}, \\
\mathcal{X}_a &= U_i \mathcal{X}_a U_i^{-1}, \quad a \neq i, \quad (a,i) \neq (1,2).
\end{aligned}$$

As before, there exists a solution on a commutative twisted torus and sets of solutions on NC deformations.

This goes through in any dimension.

The general $\{\hat{x}, \hat{\partial}\}$-algebra for any higher-dimensional nilmanifold:

$$\begin{aligned}
[\hat{x}^i, \hat{x}^j] &= iR_{(ij)} f^{ij}{}_k \hat{x}^k + 2\pi i \theta^{ij}, \\
[\hat{\partial}_i, \hat{\partial}_j] &= 0, \\
[\hat{\partial}_i, \hat{x}^j] &= \delta_i^j + iR_{(jk)} f^{jk}{}_i \hat{\partial}_k, \quad j < k \ .
\end{aligned}$$

$f_{ij}{}^k$ are the structure constants of the algebra $\mathcal{A}_d$,

and $R_{(ij)}$ are combinations of the radii.

Relations between backgrounds

Q: How are the solutions related to supergravity backgrounds?

Q: Are the solutions related among themselves?

Brief summary of solutions:

	$[\hat{x}^i, \hat{x}^j]$	Torus type	Twist
T^3	0	3d C	No
T^3_θ	$-2\pi i\theta^{ij}$	3d NC	No
$\mathrm{T}^3_\times$	$iN\epsilon_{ij2}\hat{x}^2$	3d NC,NA	No
$\tilde{\mathrm{T}}^3$	0	3d C	✓
$\tilde{\mathrm{T}}^3_{\times,\theta}$	$i\frac{R_2R_3}{R_1}\epsilon_{ij1}\hat{x}^1 + 2\pi i\theta^{ij}$	3d NC	✓
$\tilde{\mathrm{T}}^6$	0	6d C	✓
$\tilde{\mathrm{T}}^6_{\times,\theta}$	$iR_{(ij)}f^{ij}{}_k\hat{x}^k + 2\pi i\theta^{ij}$	6d NC	✓

Connes-Douglas-Schwarz conjecture

Deformation parameters θ_{ij} of MM on $\mathrm{T}^d_\theta \leftrightarrow$ moduli of 11D sugra

11D sugra: 3-form potential C_{IJK}. Claim: $\theta_{ij} \propto \int dx^i dx^j C_{ij-}$.

In IIA language: $\theta_{ij} \propto \int dx^i dx^j B_{ij}$.

$\rightsquigarrow$ Deform tori in MM $\overset{CDS}{\longleftrightarrow}$ Turn on background fluxes in sugra

Schematically: MM on $\mathrm{T}^3_\theta \quad \overset{CDS}{\longleftrightarrow} \quad$ IIA on T^3_B,

Douglas, Hull '97, Kawano, Okuyama '98

It is reasonable to suggest:

- ✔ Deforming the torus by non-constant NC $\rightarrow$ H-flux in sugra,

$$\text{MM on } \mathrm{T}^3_x \quad \overset{CDS}{\longleftrightarrow} \quad \text{IIA on } \mathrm{T}^3_H.$$

- ✔ Twist + H-flux... Lowe, Nastase, Ramgoolam '03
  Sugra with geometric + NSNS fluxes
  Kaloper, Myers '99, Hull, Reid-Edwards '05,'06

$$\text{MM on } \tilde{\mathrm{T}}^3_x \quad \overset{CDS}{\longleftrightarrow} \quad \text{IIA on } \tilde{\mathrm{T}}^3_H,$$

$$\text{MM on } \tilde{\mathrm{T}}^6_x \quad \overset{CDS}{\longleftrightarrow} \quad \text{IIA on } \tilde{\mathrm{T}}^6_H,$$

These have to be fully demonstrated by studying in detail the resulting effective action (work in progress with L. Jonke).

Seiberg-Witten maps

SW: $\exists$ a map from NCYM to conventional YM on the same space.
Seiberg, Witten '99

For T^3_θ $\exists$ mapping to T^3: $f : \hat{x}^i \to x^i - \pi i \theta^{ij} \partial_j, \quad f : \hat{\partial}_i \to \partial_i.$

$U/\hat{U}$-algebra-preserving.

f induces the SW map for the gauge theories.

With the appropriate $\star$ product expresses the NC gauge fields as functions of the commutative ones.

This construction is possible for all configurations.

T-duality

$\exists$ T-duality among T^3 and twisted $\tilde{\mathrm{T}}^3$:

$$\text{IIA on } \mathrm{T}^3_H \quad \overset{T}{\longleftrightarrow} \quad \text{IIB on } \tilde{\mathrm{T}}^3.$$

lhs: associated to MM by CDS.
rhs: associated to MM on commutative twisted torus.

$$\begin{array}{ccc} \text{IIA on } \mathrm{T}^3_H & \overset{T}{\longleftrightarrow} & \text{IIB on } \tilde{\mathrm{T}}^3 \\ \updownarrow & & \updownarrow \\ \text{MM on } \mathrm{T}^3_x & \longleftrightarrow & \text{MM on } \tilde{\mathrm{T}}^3. \end{array}$$

Diagram relates two MM configurations...
Further nvestigation needed...

Conclusions and open questions

Main message

- ✔ A large class of new Matrix Model (flux) compactifications was determined.
- ✔ Non-commutative deformations tantamount to sugra fluxes.

Main questions

- ✔ Is that just a useful book-keeping for fluxes?
  Or could unconventional compactifications be described?
  (non-geometric, winding modes,...)
- ✔ What can we learn about non-perturbative dualities?
- ✔ Are there phenomenological vacua?
- ✔ Lessons for gravity in string theory?